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The individualistic foundation of equilibrium distribution ☆

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Abstract

This paper proposes a solution concept called the *type-symmetric randomized equilibrium* (TSRE), where agents with the same type of characteristics take the same randomized choice. It is shown that this solution concept provides a micro-foundation for the macro notion of equilibrium distribution for economies and games with many agents. In particular, any Walrasian (resp. Nash) equilibrium distribution in a large economy (resp. game) is shown to be *uniquely* determined by one TSRE if the agent space is modeled by the classical Lebesgue unit interval. The relationship of TSRE with other equilibrium notions is also established.

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1. Introduction

Atomless probability spaces, particularly the Lebesgue unit interval, have been commonly used to model many agents[1] in economics. There are two approaches to deal with equilibrium concepts associated with a continuum of agents: one is the individualized microeconomic notion that specifies a choice for each agent, and the other is the distributional macroeconomic notion that is defined as a distribution on the joint space of characteristics and choices without individualistic features. Specifically, a Walrasian allocation in a large economy (resp. a pure-strategy Nash equilibrium[2] in a large game) is defined as a measurable mapping from the agent space to the consumption space (resp. the action set) that satisfies the equilibrium conditions on the micro level. A Walrasian equilibrium distribution in a large economy (resp. a Nash equilibrium distribution in a large game) is defined as a distribution on the joint space of preferences, endowments, and consumption (resp. the joint space of payoffs and actions) that satisfies equilibrium conditions on the macro level. Both approaches have been extensively studied and applied in many areas.[3]

The distributional approach has often been regarded as a reformulation of the individualized approach, since an individualized equilibrium always induces an equilibrium distribution. However, a given equilibrium distribution may not be induced by any individualized equilibrium. For instance, in the large economy in Example 1, there exists a Walrasian equilibrium distribution that cannot be induced by any Walrasian allocation; in the large game in Example 2, there is a Nash equilibrium distribution that does not correspond to any pure-strategy Nash equilibrium. Hence, in general, the notion of equilibrium distribution is not merely an equivalent reformulation of the existing notion of individualized equilibrium. To understand the distributional approach and its advantage, one needs to find its generic microeconomic counterpart. After all, without a justifiable micro-foundation, a macro-aggregate solution concept can hardly be satisfactory.

In this paper, we introduce another notion of equilibrium via the individualized approach—a *type-symmetric randomized equilibrium* (TSRE). A TSRE is a randomized equilibrium where agents with the same type of characteristics take the same randomized choice. It is not difficult to show that in any large economy or any large game, an equilibrium distribution corresponds to a TSRE. What is a surprise, however, is that if the classical Lebesgue unit interval models the agent space, one equilibrium distribution corresponds *uniquely* to one TSRE. As the latter is an

1 Milnor and Shapley (1978), and Aumann (1964) used the Lebesgue unit interval as an agent space that captures the negligible influence of an individual in a large society. An economy with a continuum of agents is also considered to be a good approximation for economies with large but finitely many agents; see, for example, Debreu and Scarf (1963), Hildenbrand (1974), Hammond (1979), Green (1984), Mas-Colell and Vives (1993), Khan and Sun (2002), and McLean and Postlewaite (2002, 2004).

2 Throughout the paper, we shall use "agent" interchangeably with "player" in the informal discussion of large games.

3 See the book Hildenbrand (1974) and the survey Khan and Sun (2002) for various references; also see Rauh (2007), Anderson and Raimondo (2008), Yannelis (2009), Acemoglu and Wolitzky (2011), Guesnerie and Jara-Moroni (2011), Duffie and Strulovici (2012), Yang and Qi (2013), Hammond (2015), and Balbus et al. (2019) for some recent applications.

equilibrium that focuses on the individual level, it indeed provides a generic micro-foundation for the macro notion of equilibrium distribution. All this evidence reveals that an equilibrium distribution is in general a reformulation of a randomized equilibrium rather than a deterministic individualized equilibrium. For the uniqueness result, we note that some properties of analytic sets in descriptive set theory play a key role.

When uncertainty is introduced at the level of individual choices, one can consider a randomized action profile and a mixed action profile. For instance, in a large economy, while a randomized allocation is a function from the agent space to the set of distributions on the consumption space, a mixed allocation is a random process from the joint agent-sample space to the consumption space that explicitly models the independence across a continuum of agents. Since TSRE, Walrasian equilibrium, randomized Walrasian equilibrium, and mixed Walrasian equilibrium are all individualized equilibria, the relationship among them is also of interest. In particular, it is natural to ask if the individual agents who choose optimized equilibrium consumption ex ante are ex post regret-free in the sense that their realized consumption allocation forms a Walrasian equilibrium allocation when all the uncertainties are resolved. To answer these questions, we work with a rich Fubini extension[4] to model the joint agent-sample spaces. We show that a mixed Walrasian equilibrium is equivalent to a randomized Walrasian equilibrium in a large economy. We also provide a link between mixed Walrasian equilibrium and (pure) Walrasian equilibrium: a mixed allocation is a mixed Walrasian equilibrium if and only if it has the ex post Walrasian property that requires the realized allocation to be part of a Walrasian equilibrium. The analogs for a generalized framework of a large game that allows heterogeneous action sets are also established. Thus, through the notion of TSRE, we obtain a complete delineation of various equilibrium concepts under the framework of a rich Fubini extension.

The rest of the paper is organized as follows. Section 2 outlines the fundamentals of our paper: In Section 2.1, we present the set-up of a large economy and two canonical solution concepts—Walrasian equilibrium and Walrasian equilibrium distribution—and provide an example showing the non-equivalence of the two concepts. In Section 2.2, we present the analog of Section 2.1 in the context of a large game. In Section 3, we introduce the notion of type-symmetric randomized equilibrium and use it to characterize the equilibrium distributions in a large economy and in a large game, respectively. Section 4 shows the relationship of TSRE with various other equilibrium notions in a large economy and in a large generalized game with heterogeneous action sets. All the proofs are given in Section 5.

2. Basics

Unless otherwise specified, a topological space—say X—as discussed in this paper is understood to be equipped with its Borel σ-algebra $\mathcal{B}(X)$, and the measurability is defined in terms of it. For a Polish (complete separable metrizable topological) space X, $\mathcal{M}(X)$ denotes the space of Borel probability measures on X endowed with the topology of weak convergence and δ_x denotes the Dirac measure at the point $x \in X$. For a probability measure $\tau \in \mathcal{M}(X \times Y)$ on the product of two Polish spaces X and Y, τ_X and τ_Y denote the marginals of τ on X and Y, respectively.

[4] A Fubini extension extends the usual product probability space and retains the Fubini property, which is used in Sun (2006) to address the issue of joint measurability and to prove the exact law of large numbers for a continuum of independent random variables. A Fubini extension is rich if it has a continuum of independent and identically distributed random variables with the uniform distribution on [0, 1]. The formal definitions are stated in Definition 12.

2.1. A large economy

We now specify some notations and terminologies for a large economy: Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space representing the space of agents, let $\mathbb{R}^n_+$ represent the commodity space, and let $\mathscr{P}_{mo}$ represent the space of monotonic preference relations on $\mathbb{R}^n_+$. We endow the space $\mathscr{P}_{mo}$ with the metric of closed convergence.[5] The characteristics of an agent consist of a preference as well as an endowment, and thus, the space of agents' characteristics is $\mathscr{P}_{mo} \times \mathbb{R}^n_+$. Let $\mathcal{B}(\mathscr{P}_{mo} \times \mathbb{R}^n_+)$ be the Borel σ-algebra on $\mathscr{P}_{mo} \times \mathbb{R}^n_+$. A *large economy* is a measurable function $\mathscr{E}$ from the agent space $(I, \mathcal{I}, \lambda)$ to the space of characteristics $\mathscr{P}_{mo} \times \mathbb{R}^n_+$ such that for each $i \in I$, $\mathscr{E}(i) = \big(\succsim_i, \boldsymbol{e}(i)\big)$, and the mean endowment $\int_I \boldsymbol{e}(i)\, \mathrm{d}\lambda(i)$ is finite and strictly positive.

Let $D(\boldsymbol{p}, \succsim, \boldsymbol{e})$ be the demand correspondence when the price vector, preference and endowment are $\boldsymbol{p}$, $\succsim$ and $\boldsymbol{e}$, respectively. That is, $D(\boldsymbol{p}, \succsim, \boldsymbol{e})$ is the set of all maximal elements under the preference $\succsim$ in the budget set $\{\boldsymbol{x} \in \mathbb{R}^n_+ \mid \boldsymbol{p} \cdot \boldsymbol{x} \le \boldsymbol{p} \cdot \boldsymbol{e}\}$. We are now ready to state two standard notions of equilibria that are commonly used in the literature on large economies. The first one, as in Aumann (1964), involves the agent space.

Definition 1 *(Walrasian equilibrium).* An allocation $\boldsymbol{f}$ of a large economy $\mathscr{E}\colon (I, \mathcal{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$ is an integrable function from $(I, \mathcal{I}, \lambda)$ to $\mathbb{R}^n_+$, and is said to be a *Walrasian allocation* under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ if

1. for λ-almost all $i \in I$, $\boldsymbol{f}(i) \in D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$;
2. $\int_I \boldsymbol{f}(i)\, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i)\, \mathrm{d}\lambda(i)$.

The pair $(\boldsymbol{f}, \boldsymbol{p})$ above is also said to be a *Walrasian equilibrium* (WE) of the economy $\mathscr{E}$.

Given a large economy $\mathscr{E}\colon (I, \mathcal{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$, we can also consider its distributional form $\lambda\mathscr{E}^{-1}$. Note that $\lambda\mathscr{E}^{-1}$ is a distribution on $\mathscr{P}_{mo} \times \mathbb{R}^n_+$ where for each Borel subset B of $\mathscr{P}_{mo} \times \mathbb{R}^n_+$, $\lambda\mathscr{E}^{-1}(B) = \lambda\big(\mathscr{E}^{-1}(B)\big) = \lambda\big(\{i \in I \mid \mathscr{E}(i) \in B\}\big)$. As in Hildenbrand (1974), we can apply the distributional approach to model a macro notion of the solution as follows.

Definition 2 *(Walrasian equilibrium distribution).* A Borel probability measure τ on the product space[6] $(\mathscr{P}_{mo} \times \mathbb{R}^n_+) \times \mathbb{R}^n_+$ is said to be a *Walrasian equilibrium distribution* (WED) under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ of $\lambda\mathscr{E}^{-1}$ (the distributional form of a large economy $\mathscr{E}$) if the following three properties hold:

1. The marginal distribution of τ on the space of characteristics $\mathscr{P}_{mo} \times \mathbb{R}^n_+$ is $\lambda\mathscr{E}^{-1}$;
2. $\tau(E_{\boldsymbol{p}}) = 1$, where $E_{\boldsymbol{p}} = \big\{(\succsim, \boldsymbol{e}, \boldsymbol{x}) \in \mathscr{P}_{mo} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \boldsymbol{x} \in D(\boldsymbol{p}, \succsim, \boldsymbol{e})\big\}$;
3. $\int_{\mathscr{P}_{mo} \times \mathbb{R}^{n,1}_+} \boldsymbol{e}\, \mathrm{d}\lambda\mathscr{E}^{-1} = \int_{\mathbb{R}^{n,2}_+} \boldsymbol{x}\, \mathrm{d}\nu$, where ν is the marginal distribution of τ on the space of commodity space (i.e., the second $\mathbb{R}^n_+$ in $(\mathscr{P}_{mo} \times \mathbb{R}^n_+) \times \mathbb{R}^n_+$).

[5] The lemma on Hildenbrand (1974, p. 98) shows that the space $\mathscr{P}_{mo}$ with the metric of closed convergence is a G_δ set in a compact metric space. By the classical Alexandroff's Lemma (see Aliprantis and Border (2006, p. 88)), $\mathscr{P}_{mo}$ is a Polish space.

[6] To distinguish these two $\mathbb{R}^n_+$, we use $\mathbb{R}^{n,k}_+$ to denote the k-th $\mathbb{R}^n_+$ in $(\mathscr{P}_{mo} \times \mathbb{R}^n_+) \times \mathbb{R}^n_+$ for $k = 1, 2$.

In a large economy $\mathscr{E}$, if $(\boldsymbol{f}, \boldsymbol{p})$ is a Walrasian equilibrium of $\mathscr{E}$, then $\lambda(\mathscr{E}, \boldsymbol{f})^{-1}$ is a Walrasian equilibrium distribution under $\boldsymbol{p}$ of $\lambda\mathscr{E}^{-1}$. That is, the joint distribution of $\mathscr{E}$ and the Walrasian allocation $\boldsymbol{f}$ is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$.

However, the converse may not hold: Given a large economy $\mathscr{E}$ and a Walrasian equilibrium distribution τ under $\boldsymbol{p}$ of $\lambda\mathscr{E}^{-1}$, there may not exist a Walrasian allocation $\boldsymbol{f}$ under $\boldsymbol{p}$ such that $\lambda(\mathscr{E}, \boldsymbol{f})^{-1} = \tau$. Below is a simple example.[7]

Example 1 *(Inconsistency between WE and WED).* Consider the following economy $\hat{\mathscr{E}}$ with two goods. Let the space of agents be the Lebesgue unit interval $([0,1], \mathcal{B}, \ell)$, where $\mathcal{B}$ is the Borel σ-algebra and ℓ is the Lebesgue measure.

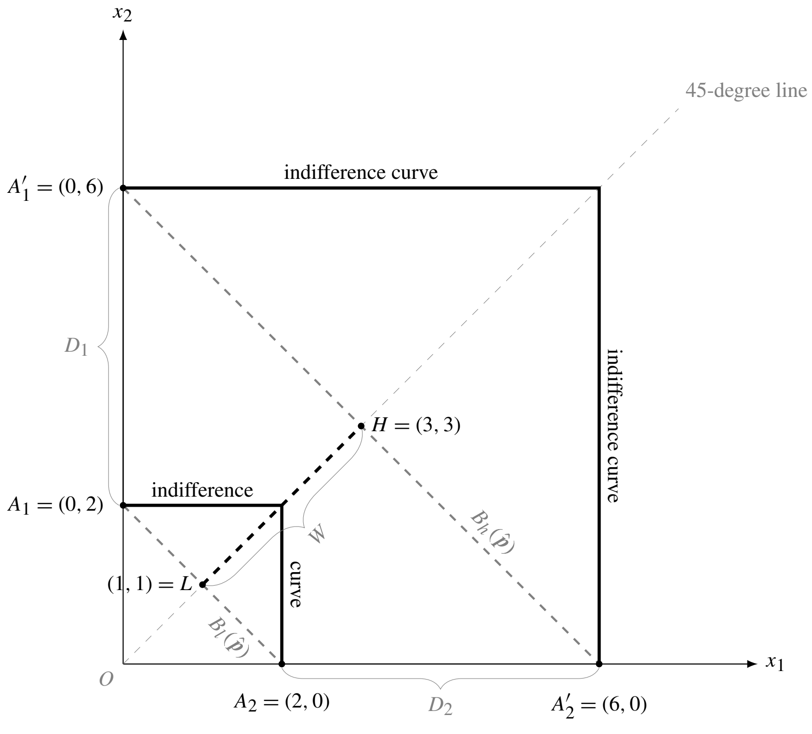
All agents share the same utility function: For each consumption bundle $(x_1, x_2) \in \mathbb{R}^2_+$,

$$u(x_1, x_2) = \max\{x_1, x_2\}.$$

The endowment of each agent $i \in [0,1]$ is

$$\boldsymbol{e}(i) = (2i + 1, 2i + 1).$$

The economy $\hat{\mathscr{E}}$ specified by the utility and the endowment is a well-defined large economy.



[7] Kannai (1970, Section 7) constructed a different example showing that the set of distributions of the Walrasian allocations is not completely determined by the distribution of agents' characteristics in terms of their preferences and endowments. In contrast to the possible nonexistence of Nash equilibrium for a large game with a one-dimensional action space as mentioned in Footnote 8, a Walrasian allocation always exists in a finite-dimensional commodity space. On the other hand, when one works with an infinite-dimensional commodity space, a Walrasian allocation may not exist; see Tourky and Yannelis (2001) for example.

The utility and endowment are illustrated in the figure above. All agents have the same utility where the corresponding indifference curves are parallel. The set of all endowments is just the line segment W: $x_1 = x_2$ for $x_1 \in [1, 3]$ with the end points $L = (1, 1)$ and $H = (3, 3)$. The line segment D_1 is represented by $x_1 = 0$ for $x_2 \in [2, 6]$ with the end points $A_1 = (0, 2)$ and $A_1' = (0, 6)$, and the line segment D_2 is represented by $x_2 = 0$ for $x_1 \in [2, 6]$ with the end points $A_2 = (2, 0)$ and $A_2' = (6, 0)$.

Let $\hat{\boldsymbol{p}} = (1, 1)$. Let $\hat{\boldsymbol{f}}_1$ and $\hat{\boldsymbol{f}}_2$ be two allocations of $\hat{\mathscr{E}}$ such that $\hat{\boldsymbol{f}}_1(i) = (0, 4i + 2)$ and $\hat{\boldsymbol{f}}_2(i) = (4i + 2, 0)$ for all $i \in [0, 1]$. Note that allocations $(0, 4i + 2)$ and $(4i + 2, 0)$ are the best choices for each agent $i \in [0, 1]$. Let $\hat{\tau} = \frac{1}{2}\ell(\hat{\mathscr{E}}, \hat{\boldsymbol{f}}_1)^{-1} + \frac{1}{2}\ell(\hat{\mathscr{E}}, \hat{\boldsymbol{f}}_2)^{-1}$. It is clear that $\hat{\tau}$ is a Walrasian equilibrium distribution of $\ell(\hat{\mathscr{E}})^{-1}$ under $\hat{\boldsymbol{p}}$.

However, there does not exist a Walrasian allocation $\hat{\boldsymbol{f}}$ under $\hat{\boldsymbol{p}} = (1, 1)$ such that $\ell(\hat{\mathscr{E}}, \hat{\boldsymbol{f}})^{-1} = \hat{\tau}$. To see this, fix any Walrasian equilibrium $(\hat{\boldsymbol{f}}, \hat{\boldsymbol{p}})$ in the economy $\hat{\mathscr{E}}$. Assume that $\ell(\hat{\mathscr{E}}, \hat{\boldsymbol{f}})^{-1} = \hat{\tau}$. Then, $\hat{\boldsymbol{f}}$ induces the uniform distribution μ on the set $D_1 \cup D_2$. It is clear that the best response of agent $i \in [0, 1]$ is the intersection of the budget line and the set $D_1 \cup D_2$. That is, the best response correspondence is

$$\hat{\boldsymbol{F}}(i) = \big\{(0, 4i + 2), (4i + 2, 0)\big\} \text{ for each } i \in [0, 1].$$

Since $\hat{\boldsymbol{f}}$ is a Walrasian allocation in the economy $\hat{\mathscr{E}}$ under $\hat{\boldsymbol{p}} = (1, 1)$, we have $\hat{\boldsymbol{f}}(i) \in \hat{\boldsymbol{F}}(i)$ for ℓ-almost all $i \in [0, 1]$. For $j = 1, 2$, let $C_j = \hat{\boldsymbol{f}}^{-1}(D_j)$. Then $\hat{\boldsymbol{f}}(C_j) \subseteq D_j$ for $j = 1, 2$. Since $\hat{\boldsymbol{f}}$ induces the uniform distribution μ on the set $D_1 \cup D_2$, we know that $\mu\big(\hat{\boldsymbol{f}}(C_j)\big) = \ell(C_j)$. Now, consider the case $j = 1$. Since μ is the uniform distribution on $D_1 \cup D_2$ and $\hat{\boldsymbol{f}}(i) = (0, 4i + 2)$ for any $i \in C_1$, we can obtain that $\mu\big(\hat{\boldsymbol{f}}(C_1)\big) = \frac{\ell(C_1)}{2}$. Similarly, we have $\mu\big(\hat{\boldsymbol{f}}(C_2)\big) = \frac{\ell(C_2)}{2}$. Therefore, for each $j = 1, 2$, $\ell(C_j) = \frac{\ell(C_j)}{2}$, which implies that $\ell(C_j) = 0$. This is a contradiction. Therefore, there does not exist any Walrasian allocation $\hat{\boldsymbol{f}}$ under $\hat{\boldsymbol{p}}$ such that $\ell(\hat{\mathscr{E}}, \hat{\boldsymbol{f}})^{-1} = \hat{\tau}$. □

2.2. A large game

We next introduce some specific notation and terminology for a conventional large game as surveyed in Khan and Sun (2002): Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space representing the space of players and a compact metric space A representing a common action space. Let $\mathcal{M}(A)$ denote the set of all (Borel) probability distributions on A. An element of $\mathcal{M}(A)$, as an action distribution, represents a possible externality or a societal summary. Let the space of players' characteristics $\mathcal{U}_A$ be given by the space of all continuous functions on the product space $A \times \mathcal{M}(A)$. Endowed with its sup-norm topology and the resulting Borel σ-algebra, $\mathcal{U}_A$ would be conceived as a measurable space $\big(\mathcal{U}_A, \mathcal{B}(\mathcal{U}_A)\big)$. A *large game* $\mathcal{G}$, as conventionally defined, is a measurable function on $(I, \mathcal{I}, \lambda)$ taking values in $\mathcal{U}_A$. Given a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, one can consider its distributional form $\lambda\mathcal{G}^{-1}$, which is a Borel probability measure on $\mathcal{U}_A$.

Similar to the solution concepts in a large economy, we also have both individual and distributional approaches to define an equilibrium in a large game. The individual approach involves the player space.

Definition 3 *(Pure-strategy Nash equilibrium).* A pure strategy profile f of a large game $\mathcal{G}\colon I \to \mathcal{U}_A$ is a measurable function from $(I, \mathcal{I}, \lambda)$ to the action set A, and is said to be a *pure-strategy Nash equilibrium* (pure NE) if for λ-almost all $i \in I$,

$$u_i\big(f(i), \lambda f^{-1}\big) \geq u_i\big(a, \lambda f^{-1}\big) \text{ for all } a \in A,$$

with u_i abbreviated for $\mathcal{G}(i)$.

Ignoring the player space, we can consider the distributional form $\lambda\mathcal{G}^{-1}$ of a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, with its corresponding solution concept defined as follows.

Definition 4 *(Nash equilibrium distribution).* A Borel probability measure τ on $\mathcal{U}_A \times A$ is said to be a *Nash equilibrium distribution* (NED) of $\lambda\mathcal{G}^{-1}$ (the distributional form of a large game $\mathcal{G}$) if $\tau_{\mathcal{U}_A} = \lambda\mathcal{G}^{-1}$ and $\tau\big(\mathrm{Br}(\tau)\big) = 1$ where

$$\mathrm{Br}(\tau) = \big\{(u, a) \in \mathcal{U}_A \times A \mid u(a, \tau_A) \geq u(x, \tau_A) \text{ for all } x \in A\big\}.$$

It is now well understood that in a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, if f is a pure-strategy Nash equilibrium of $\mathcal{G}$, then $\lambda(\mathcal{G}, f)^{-1}$ is a Nash equilibrium distribution of $\lambda\mathcal{G}^{-1}$. Hence, given any pure-strategy Nash equilibrium as a solution at the individual microeconomic level, its macroeconomics counterpart automatically becomes a Nash equilibrium distribution of the same game in its distributional form.

However, the converse often fails. Whereas a Nash equilibrium distribution always exists, when the action set is uncountable, there are examples where a pure-strategy Nash equilibrium does not exist at all; see Footnote 8. Thus, in a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, given a Nash equilibrium distribution τ of $\lambda\mathcal{G}^{-1}$, it is possible that there does not exist any pure-strategy Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda(\mathcal{G}, f)^{-1}$. This statement also holds when both pure-strategy Nash equilibrium and Nash equilibrium distribution exist in a large game. Below is a simple example.[8]

Example 2 *(Inconsistency between pure NE and NED).* Let the player space be the Lebesgue unit interval $([0, 1], \mathcal{B}, \ell)$ and the common action set be $\hat{A} = [0, 1]$. Let $\hat{\mathcal{G}}\colon [0, 1] \to \mathcal{U}_{\hat{A}}$ be such that for each player $i \in [0, 1]$, each action $a \in \hat{A}$, and each societal play $\mu \in \mathcal{M}(\hat{A})$,

$$\hat{\mathcal{G}}(i)(a, \mu) = -a^2 \cdot (a - i)^2.$$

It is clear that $\hat{\mathcal{G}}$ is a well-defined large game. Let $\hat{f}_1$ and $\hat{f}_2$ be two pure strategy profiles of $\hat{\mathcal{G}}$ such that $\hat{f}_1(i) = 0$ and $\hat{f}_2(i) = i$ for each $i \in [0, 1]$. As actions 0 and i are the best choices for player i, $\hat{f}_1$ and $\hat{f}_2$ are pure-strategy Nash equilibria of $\hat{\mathcal{G}}$. Let $\hat{\tau} = \frac{1}{2}\ell(\hat{\mathcal{G}}, \hat{f}_1)^{-1} + \frac{1}{2}\ell(\hat{\mathcal{G}}, \hat{f}_2)^{-1}$. It is easy to check that $\hat{\tau}$ is a Nash equilibrium distribution of $\ell(\hat{\mathcal{G}})^{-1}$.

However, there is no pure-strategy Nash equilibrium $\hat{f}$ such that $\ell(\hat{\mathcal{G}}, \hat{f})^{-1} = \hat{\tau}$. Suppose not. Then by the construction of $\hat{\tau}$, we must have $\ell\hat{f}^{-1} = \frac{1}{2}\delta_0 + \frac{1}{2}\ell$. Furthermore, since $\hat{f}$ is a pure-strategy Nash equilibrium, $\hat{f}(i)$ should be 0 or i for ℓ-almost all $i \in [0, 1]$. Suppose that $\hat{f}(i) = i$ holds on some set $C \in \mathcal{B}$. Then, we must have

$$\ell(C) = \ell\hat{f}^{-1}(C) = \big(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\ell\big)(C),$$

and hence, $\ell(C)$ is 0 or 1. This finding contradicts the fact that $\ell\hat{f}^{-1} = \frac{1}{2}\delta_0 + \frac{1}{2}\ell$. □

[8] In this example, a Nash equilibrium and a Nash equilibrium distribution exist. Rath et al. (1995) presented a rather involved example of a large game with the action space $[-1, 1]$ that has a Nash equilibrium distribution but no Nash equilibrium; see also Khan et al. (2013) and Qiao and Yu (2014) for some recent examples of this kind. In consideration of the existence issue, Keisler and Sun (2009), Khan et al. (2013) and He et al. (2017) showed that the (relative) saturation of the agent space is necessary and sufficient for the existence of a Nash equilibrium; see also Barelli and Duggan (2015) for the agent space with a product structure.

3. Type-symmetric randomized equilibrium

As seen in Examples 1 and 2, the usual notion of individualized equilibrium may not be an ideal counterpart to equilibrium distribution. In this section, we formally introduce the notion of type-symmetric randomized equilibrium (TSRE), which is a randomized equilibrium where agents with the same type of characteristics take the same randomized choice. We illustrate such a notion in both large economies and large games. It is shown that this new notion of individualized equilibrium provides a generic micro-foundation of equilibrium distribution (WED or NED) in the sense a TSRE corresponds to an equilibrium distribution and vice versa. More importantly, if the agent space is modeled by the classical Lebesgue unit interval, one equilibrium distribution corresponds *uniquely* to one TSRE, which means that an equilibrium distribution is in general a reformulation of a randomized equilibrium rather than a deterministic individualized equilibrium.

3.1. Type-symmetric randomized equilibrium in a large economy

We now follow the notation in Section 2.1 and present the definition of type-symmetric randomized equilibrium in large economies.

Definition 5 *(TSRE in large economy).* A randomized allocation $\boldsymbol{h}$ of a large economy $\mathscr{E}$ is a measurable function[9] from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(\mathbb{R}^n_+)$, and is said to be a *randomized Walrasian allocation* under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ if

1. for λ-almost all $i \in I$, $\boldsymbol{h}(i) \in \mathcal{M}(\mathbb{R}^n_+)$ with support in $D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$;
2. $\int_I \int_{\mathbb{R}^n_+} \boldsymbol{x}\boldsymbol{h}(i; \mathrm{d}\boldsymbol{x})\, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i)\, \mathrm{d}\lambda(i)$.

The pair $(\boldsymbol{h}, \boldsymbol{p})$ above is also said to be a *randomized Walrasian equilibrium* (randomized WE) of $\mathscr{E}$. A randomized Walrasian equilibrium $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$ is said to be a *type-symmetric randomized equilibrium* (TSRE) if $\boldsymbol{h}(i) = \boldsymbol{h}(j)$ whenever $\mathscr{E}(i) = \mathscr{E}(j)$.

In Definition 5, Condition 1 indicates that almost every agent's randomized equilibrium consumption assigns zero probability for the set of non-optimal choices. Condition 2 is the market clearing condition. Also, as noted in Footnote 9, it is easy to see that a randomized strategy profile $\boldsymbol{h}$ naturally induces the societal consumption distribution $\int_I \boldsymbol{h}(i)\, \mathrm{d}\lambda(i)$ where $\big(\int_I \boldsymbol{h}(i)\, \mathrm{d}\lambda(i)\big)(B) = \int_I \boldsymbol{h}(i; B)\, \mathrm{d}\lambda(i)$ for each $B \in \mathcal{B}(\mathbb{R}^n_+)$. Similarly, a large economy $\mathscr{E}$ and a randomized allocation $\boldsymbol{h}$ of $\mathscr{E}$ also naturally induce a joint *type-consumption distribution* $\int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\, \mathrm{d}\lambda(j)$ as the corresponding macro aggregate, where $\big(\int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\, \mathrm{d}\lambda(j)\big)(E) = \int_I \big(\delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\big)(E)\, \mathrm{d}\lambda(j)$ for each $E \in \mathcal{B}(\mathscr{P}_{mo} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+)$.

A TSRE is a refinement of a randomized Walrasian equilibrium in the sense that all agents with the same type of characteristics are required to choose the *same randomized allocation*. We next show that this new notion of equilibrium not only provides a generic micro-foundation of Walrasian equilibrium distribution in general but also presents a uniqueness characterization of a Walrasian equilibrium distribution when the agent space is the Lebesgue unit interval $([0, 1], \mathcal{B}, \ell)$.

[9] A measurable function $h\colon I \to \mathcal{M}(X)$, where X is a Polish space, can also be viewed as a transition probability $h\colon I \times \mathcal{B}(X) \to [0, 1]$ such that (i) for every $B \in \mathcal{B}(X)$, $h(\cdot; B)$ is measurable, (ii) for λ-almost all $i \in I$, $h(i; \cdot) \in \mathcal{M}(X)$; see Remark 3.20 in Crauel (2002) for example.

Theorem 1. *In a large economy $\mathscr{E}: (I, \mathcal{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$, we have the following statements:*

(i) For any TSRE $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$, the joint type-consumption distribution induced by $\boldsymbol{h}$, $\int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\,\mathrm{d}\lambda(j)$, is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$.
(ii) For any Walrasian equilibrium distribution τ of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, there is a TSRE $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$ such that $\boldsymbol{h}$ induces it, i.e., $\int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\,\mathrm{d}\lambda(j) = \tau$.
(iii) If $(I, \mathcal{I}, \lambda)$ is the Lebesgue unit interval, then for any Walrasian equilibrium distribution τ of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, there is a unique *TSRE allocation $\boldsymbol{h}$ under $\boldsymbol{p}$ such that $\boldsymbol{h}$ induces it.*

In the above theorem, statement (i) basically suggests that the type-consumption distribution associated with any TSRE is a Walrasian equilibrium distribution, and statement (ii) says that every Walrasian equilibrium distribution can be lifted to some TSRE at the individual level so that the TSRE allocation induces the same aggregate as given. These characterizations themselves are new in the literature. Statement (iii) in Theorem 1 is surprising, as it says that any Walrasian equilibrium distribution τ is *uniquely* determined by a particular TSRE allocation $\boldsymbol{h}$ when the agent space is the Lebesgue unit interval. That is to say, for large economies with the Lebesgue agent space, we have a one-to-one characterization between an equilibrium distribution at the aggregate level and an equilibrium at the individual level.

We now return to Example 1 and consider $\hat{\boldsymbol{h}}: [0, 1] \to \mathcal{M}(\mathbb{R}^2_+)$, a randomized allocation of $\hat{\mathscr{E}}$ such that for each $i \in [0, 1]$,

$$\hat{\boldsymbol{h}}(i) = \tfrac{1}{2}\delta_{(0,4i+2)} + \tfrac{1}{2}\delta_{(4i+2,0)}. \tag{1}$$

It is clear that $(\hat{\boldsymbol{h}}, \hat{\boldsymbol{p}})$ is a TSRE where $\hat{\boldsymbol{h}}$ induces the Walrasian equilibrium distribution $\hat{\tau}$ given in Example 1. Furthermore, since a Walrasian equilibrium is a degenerated randomized Walrasian equilibrium and every agent in $\hat{\mathscr{E}}$ in Example 1 has an endowment different from others, any Walrasian equilibrium of $\hat{\mathscr{E}}$ is a degenerated TSRE. Statement (iii) in Theorem 1 implies that $\hat{\boldsymbol{h}}$ is the unique TSRE allocation (under $\hat{\boldsymbol{p}}$) that induces $\hat{\tau}$. Thus, we show directly that there is no Walrasian equilibrium of $\hat{\mathscr{E}}$ that can induce $\hat{\tau}$, and we reemphasize that whereas Walrasian equilibrium is not suitable as a micro-foundation for Walrasian equilibrium distribution, our new notion of equilibrium, TSRE, is indeed a generic and sharp micro counterpart to Walrasian equilibrium distribution.

We use the example below to highlight that Theorem 1 (iii) may fail to hold if a probability space other than the Lebesgue unit interval is used as the agent space.

Example 3. Consider an economy $\bar{\mathscr{E}}$ that differs from the economy $\hat{\mathscr{E}}$ defined in Example 1 only in terms of the agent space: let that of $\bar{\mathscr{E}}$ be $([0, 1], \bar{\mathcal{B}}, \bar{\ell})$—an extension of the Lebesgue unit interval $([0, 1], \mathcal{B}, \ell)$ such that there is a $\bar{\mathcal{B}}$-measurable[10] subset B of measure half and independent with $\mathcal{B}$ under $\bar{\ell}$. It is obvious that $\hat{\tau}$ defined in Example 1 is still a Walrasian equilibrium distribution of $\bar{\ell}(\bar{\mathscr{E}})^{-1}$ and that $\hat{\boldsymbol{h}}$ defined as in Equation (1) is a TSRE allocation such that $(\bar{\mathscr{E}}, \hat{\boldsymbol{h}})$ induces the Walrasian equilibrium distribution $\hat{\tau}$.

[10] It is well-known that there is a nonmeasurable (in the Lebesgue sense) subset B in $[0, 1]$ with the inner measure zero and the outer measure one; see the construction in Section 2.7 in Loeb (2016). Let $\bar{\mathcal{B}} = \{(B_1 \cap B) \cup (B_2 \setminus B) \mid B_1, B_2 \in \mathcal{B}\}$ and $\bar{\ell}\big((B_1 \cap B) \cup (B_2 \setminus B)\big) = \frac{1}{2}\big(\ell(B_1) + \ell(B_2)\big)$ for each B_1 and B_2 in $\mathcal{B}$. It is easy to check that $([0, 1], \bar{\mathcal{B}}, \bar{\ell})$ is a probability space and the set B is of measure $\frac{1}{2}$ and independent with $\mathcal{B}$ under $\bar{\ell}$.

Let $\bar{\boldsymbol{h}}\colon [0,1] \to \mathcal{M}(\mathbb{R}^2_+)$ be defined as follows:

$$\bar{\boldsymbol{h}}(i) = \begin{cases} \delta_{(0,4i+2)}, & \text{if } i \in B, \\ \delta_{(4i+2,0)}, & \text{if } i \in [0,1] \setminus B. \end{cases}$$

It is clear that $\bar{\boldsymbol{h}}$ is a TSRE allocation of $\bar{\mathscr{E}}$ and induces the Walrasian equilibrium distribution $\hat{\tau}$. Thus, the uniqueness result in Theorem 1 (iii) does not work for $\bar{\mathscr{E}}$. □

3.2. *Type-symmetric randomized equilibrium in a large game*

We now follow the notation in Section 2.2, and present the definition of type-symmetric randomized equilibrium in large games.

Definition 6 *(TSRE in large games).* A randomized strategy profile h of a large game $\mathcal{G}\colon I \to \mathcal{U}_A$ is a measurable function from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(A)$, and is said to be a *randomized-strategy Nash equilibrium* (randomized NE) if, for λ-almost all $i \in I$,

$$\int_A u_i\Big(a, \int_I h(j)\,\mathrm{d}\lambda(j)\Big) h(i; \mathrm{d}a) \geq \int_A u_i\Big(a, \int_I h(j)\,\mathrm{d}\lambda(j)\Big)\,\mathrm{d}\eta(a) \tag{2}$$

for all $\eta \in \mathcal{M}(A)$. A randomized-strategy Nash equilibrium h of $\mathcal{G}$ is said to be a *type-symmetric randomized equilibrium* (TSRE) if $h(i) = h(j)$ whenever $\mathcal{G}(i) = \mathcal{G}(j)$.

A randomized strategy profile $h\colon (I, \mathcal{I}, \lambda) \to \mathcal{M}(A)$ can be viewed as a transition probability $h\colon I \times \mathcal{B}(A) \to [0,1]$, as noted in Footnote 9. Thus, a randomized strategy profile h naturally induces the societal action distribution $\int_I h(j)\,\mathrm{d}\lambda(j)$ where for each $B \in \mathcal{B}(A)$, $\big(\int_I h(j)\,\mathrm{d}\lambda(j)\big)(B) = \int_I h(j; B)\,\mathrm{d}\lambda(j)$. In Definition 6, Equation (2) requires the randomized equilibrium strategies to be optimal with respect to the societal action distribution $\int_I h(j)\,\mathrm{d}\lambda(j)$ in terms of the expected payoffs. Note that for any randomized strategy profile h of a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, the corresponding *type-action distribution* $\int_I \delta_{\mathcal{G}(j)} \otimes h(j)\,\mathrm{d}\lambda(j)$, where for each $E \in \mathcal{B}(\mathcal{U}_A \times A)$, $\big(\int_I \delta_{\mathcal{G}(j)} \otimes h(j)\,\mathrm{d}\lambda(j)\big)(E) = \int_I \delta_{\mathcal{G}(j)} \otimes h(j)(E)\,\mathrm{d}\lambda(j)$, is a well-defined macro aggregate distribution on $\mathcal{U}_A \times A$.

It is also worth pointing out that every pure strategy profile f naturally corresponds to the randomized strategy profile h^f where $h^f(i) = \delta_{f(i)}$, i.e., $h^f(i)(\{f(i)\}) = 1$, for each $i \in I$. Therefore, we have that for every $B \in \mathcal{B}(A)$,

$$\begin{aligned}\Big(\int_I h^f(j)\,\mathrm{d}\lambda(j)\Big)(B) &= \Big(\int_I \delta_{f(j)}\,\mathrm{d}\lambda(j)\Big)(B) = \int_I \delta_{f(j)}(B)\,\mathrm{d}\lambda(j) \\ &= \lambda\big(\{j \in I \mid f(j) \in B\}\big) = \lambda f^{-1}(B).\end{aligned}$$

If h^f is a randomized-strategy Nash equilibrium, then Equation (2) becomes

$$u_i\big(f(i), \lambda f^{-1}\big) \geq \int_A u_i(a, \lambda f^{-1})\,\mathrm{d}\eta(a)$$

for all $\eta \in \mathcal{M}(A)$, which coincides with Definition 3 and suggests that f is a pure-strategy Nash equilibrium.

We next present an analog of Theorem 1 in the context of a large game.

Theorem 2. *In a large game $\mathcal{G}\colon I \to \mathcal{U}_A$, we have the following statements:*

(i) For any TSRE of $\mathcal{G}$, the induced type-action distribution is a Nash equilibrium distribution of $\lambda\mathcal{G}^{-1}$.
(ii) For any Nash equilibrium distribution of $\lambda\mathcal{G}^{-1}$, there is a TSRE of $\mathcal{G}$ that induces it.
(iii) If $(I, \mathcal{I}, \lambda)$ is the Lebesgue unit interval, then for any Nash equilibrium distribution of $\lambda\mathcal{G}^{-1}$, there is a unique *TSRE of $\mathcal{G}$ that induces it.*

The result above shows that in a large game, the notion of TSRE also provides not only a generic micro-foundation of Nash equilibrium distribution in general but also a uniqueness characterization of a Nash equilibrium distribution when the player space is the Lebesgue unit interval $([0, 1], \mathcal{B}, \ell)$. We now get back to Example 2 and consider $\hat{h}\colon [0, 1] \to \mathcal{M}(\hat{A})$, a randomized strategy profile of the large game $\hat{\mathcal{G}}$, such that for each $i \in [0, 1]$,

$$\hat{h}(i) = \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_i. \tag{3}$$

By definition, $\hat{h}$ is a TSRE that induces the Nash equilibrium distribution $\hat{\tau}$ given in Example 2. As a pure-strategy Nash equilibrium is a degenerated randomized-strategy Nash equilibrium and every player in $\hat{\mathcal{G}}$ in Example 2 has a payoff different from others, any pure-strategy Nash equilibrium of $\hat{\mathcal{G}}$ can be viewed as a (degenerated) TSRE. Statement (iii) in Theorem 2 implies that $\hat{h}$ is the unique TSRE that induces $\hat{\tau}$. Thus, we show directly that there is no pure-strategy Nash equilibrium of $\hat{\mathcal{G}}$ that can induce $\hat{\tau}$ and reemphasize that while pure-strategy Nash equilibrium is not suitable as a micro-foundation for Nash equilibrium distribution, the new notion of equilibrium, TSRE, is instead the micro counterpart to Nash equilibrium distribution.

Theorem 2 (iii) may fail to hold if an atomless probability space other than the Lebesgue unit interval is used as the player space. Below is an example.

Example 4. Consider a game $\bar{\mathcal{G}}$ that is almost the same as the game $\hat{\mathcal{G}}$ in Example 2 except the player space: let that of $\bar{\mathcal{G}}$ be $([0, 1], \bar{\mathcal{B}}, \bar{\ell})$, the extension of the Lebesgue unit interval $([0, 1], \mathcal{B}, \ell)$ used in Example 3. Note that $\hat{\tau}$ defined in Example 2 is still a Nash equilibrium distribution of $\bar{\ell}(\bar{\mathcal{G}})^{-1}$. Furthermore, it is clear that $\hat{h}$ defined as in Equation (3) is a TSRE of $\bar{\mathcal{G}}$, and $(\bar{\mathcal{G}}, \hat{h})$ induces the Nash equilibrium distribution $\hat{\tau}$.

Let $\bar{h}\colon [0, 1] \to \mathcal{M}(\hat{A})$ be such that

$$\bar{h}(i) = \begin{cases} \delta_i, & \text{if } i \in B, \\ \delta_0, & \text{if } i \in [0, 1] \setminus B. \end{cases}$$

While it is easy to see that $\bar{h}$ is a TSRE of the game $\bar{\mathcal{G}}$ which is different as $\hat{h}$, $\bar{h}$ also induces the Nash equilibrium distribution $\hat{\tau}$. Thus, the uniqueness characterization in Theorem 2 (iii) fails in the game $\bar{\mathcal{G}}$. □

3.3. Type-symmetric randomized equilibrium and symmetrization

In the subsection, we first focus on a large game and make two observations related to its TSRE. In the discussion immediately following Equation (3) above, we have used the type-symmetric idea to pure-strategy Nash equilibria since a pure-strategy Nash equilibrium can be always viewed as a randomized-strategy Nash equilibrium. We next make this refinement of Nash equilibrium explicit and say that a pure strategy profile f^s of a large game $\mathcal{G}\colon I \to \mathcal{U}_A$

is a *type-symmetric pure-strategy Nash equilibrium* if f^s is a pure-strategy Nash equilibrium and $f^s(i) = f^s(j)$ whenever $\mathcal{G}(i) = \mathcal{G}(j)$. This refinement explicitly connects the symmetric individual behavior and the societal outcome and allows us to characterize a symmetric Nash equilibrium distribution, a macro solution concept that implicitly requires that all players with the same type of characteristics play the same action; see Mas-Colell (1984) and Khan and Sun (2002, Section 4). Note that a Nash equilibrium distribution τ of $\lambda\mathcal{G}^{-1}$ is said to be *symmetric* if there exists a measurable function $s: \mathcal{U}_A \to A$ such that $\tau(\text{graph of } s) = 1$. Below is a direct corollary of Theorem 2.

Corollary 1. *In a large game $\mathcal{G}$ with the Lebesgue player space* $([0,1], \mathcal{B}([0,1]), \ell)$*, if τ is a symmetric Nash equilibrium distribution, there exists a unique type-symmetric pure-strategy Nash equilibrium f such that $\ell(\mathcal{G}, f)^{-1} = \tau$.*

Note that it is clear that the notion of TSRE refines the notion of randomized-strategy Nash equilibrium. We next connect a randomized-strategy Nash equilibrium to a TSRE in a large game. We say that a randomized-strategy Nash equilibrium h can be *symmetrized* if there exists a TSRE h^s such that h and h^s induce the same type-action distribution.

Corollary 2. *Every randomized-strategy Nash equilibrium in a large game can be symmetrized.*

This finding suggests that there is no essential difference between a randomized-strategy Nash equilibrium and a TSRE at the aggregate level.

We now turn to a large economy $\mathscr{E}: I \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$. Note that the symmetric allocation has also been used in a large economy; see the references of equal treatment and the discussion of a symmetric mechanism in Hammond (1979). Similar to the discussion above for a large game, the symmetric refinements of Walrasian allocations and Walrasian equilibrium distributions can be defined accordingly for a large economy. Specifically, we say that in $\mathscr{E}: I \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$, a Walrasian allocation $\boldsymbol{f}^s$ is *type-symmetric* if $\boldsymbol{f}^s(i) = \boldsymbol{f}^s(j)$ whenever $\mathscr{E}(i) = \mathscr{E}(j)$, and a Walrasian equilibrium distribution τ of $\lambda\mathscr{E}^{-1}$ is *symmetric* if there exists a measurable function $s: \mathscr{P}_{mo} \times \mathbb{R}^{n,1}_+ \to \mathbb{R}^{n,2}_+$ such that $\tau(\text{graph of } s) = 1$. We say that a randomized Walrasian equilibrium $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$ can be *symmetrized* if there exists a TSRE allocation $\boldsymbol{h}^s$ under $\boldsymbol{p}$ such that $\boldsymbol{h}$ and $\boldsymbol{h}^s$ induce the same type-consumption distribution. Below are the analogs of Corollaries 1 and 2 in a large economy.

Corollary 3. *In a large economy $\mathscr{E}$ with the Lebesgue agent space* $([0,1], \mathcal{B}, \ell)$*, if τ is a symmetric Walrasian equilibrium distribution under $\boldsymbol{p}$, then there exists a unique type-symmetric Walrasian allocation $\boldsymbol{f}$ under $\boldsymbol{p}$ such that $\ell(\mathscr{E}, \boldsymbol{f})^{-1} = \tau$.*

Corollary 4. *Every randomized Walrasian equilibrium in a large economy can be symmetrized.*

4. Relationship of various equilibrium notions

We have demonstrated that in a large economy or a large game, an equilibrium distribution corresponds to a type-symmetric randomized equilibrium, and this correspondence is unique when the agent space is the Lebesgue unit interval. The uniqueness characterization explains that the robustness of the existence of an equilibrium distribution simply lies in that it is essentially a notion of randomized equilibrium. In this section, we focus on the relationship among type-symmetric randomized equilibrium and other equilibrium notions in the mixed or deterministic settings. As a mixed allocation or a mixed strategy profile requires the randomization to be independent across agents, it leads to a process with a continuum of independent random variables in the setting of a continuum of agents. In order to resolve the measurability issues[11] of these processes and to guarantee the existence of these processes with a variety of distributions, we adopt the framework of a rich Fubini extension as in Sun (2006).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space that captures all the uncertainty associated with the individual randomization for the agents in a mixed allocation or a mixed strategy profile. Throughout the rest of this section, whenever we deal with a mixed allocation or mixed strategy profile, we will also assume that the agent space $(I, \mathcal{I}, \lambda)$ together with the sample space $(\Omega, \mathcal{F}, \mathbf{P})$ allows a rich Fubini extension.[12] Recall that a *Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ is a probability space that extends the usual product space of the agent space $(I, \mathcal{I}, \lambda)$ and a sample space $(\Omega, \mathcal{F}, \mathbf{P})$, and retains the Fubini property. Such a Fubini extension is *rich* if there is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process F from $I \times \Omega$ to $[0, 1]$ such that the random variables $F_i(\cdot) = F(i, \cdot)$ are independent and have the uniform distribution on $[0, 1]$. A process F from a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to a Polish space X is said to be *essentially pairwise independent* if for λ-almost all $i \in I$, the random variables F_i and F_j are independent for λ-almost all $j \in I$.[13]

4.1. Equilibrium notions in a large economy

To study mixed Walrasian equilibria in a large economy, we use a utility function u_i to represent agent i's preference $\succsim_i$ for each $i \in I$. Let $\mathcal{U}_{mo}$ be the space of monotonic functions on $\mathbb{R}^n_+$. Then, a large economy is a measurable function $\mathscr{E}$ from the agent space $(I, \mathcal{I}, \lambda)$ to the space of characteristics $\mathcal{U}_{mo} \times \mathbb{R}^n_+$ such that for each $i \in I$, $\mathscr{E}(i) = (u_i, \boldsymbol{e}_i)$.

Definition 7 *(Mixed Walrasian equilibrium).* A mixed allocation of a large economy $\mathscr{E}$ is a measurable function $\boldsymbol{g}\colon I \times \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{g}$ is essentially pairwise independent. A mixed allocation g is said to be a *mixed Walrasian allocation* of $\mathscr{E}$ under a non-zero price vector $\boldsymbol{p}$ if the following two conditions hold.

1. For λ-almost all $i \in I$ and for $\mathbf{P}$-almost all $\omega \in \Omega$, $\boldsymbol{g}_i(\omega) \cdot \boldsymbol{p} \le \boldsymbol{e}(i) \cdot \boldsymbol{p}$. For λ-almost all $i \in I$, $\int_\Omega u_i(\boldsymbol{g}_i(\omega))\, \mathrm{d}\mathbf{P}(\omega) \ge \int_\Omega u_i(\boldsymbol{g}'(\omega))\, \mathrm{d}\mathbf{P}(\omega)$ for each $\mathbf{P}$-measurable function $\boldsymbol{g}'\colon \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{g}'(\omega) \cdot \boldsymbol{p} \le \boldsymbol{e}(i) \cdot \boldsymbol{p}$ for $\mathbf{P}$-almost all $\omega \in \Omega$.
2. For $\mathbf{P}$-almost all $\omega \in \Omega$, $\int_I \boldsymbol{g}_\omega(i)\, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i)\, \mathrm{d}\lambda(i)$, where $\boldsymbol{g}_\omega(i)$ denotes $\boldsymbol{g}(i, \omega)$.

[11] See Footnote 4; also see Khan et al. (2015) for a discussion of the issues in modeling mixed-strategy Nash equilibrium in a large game.

[12] The usual Lebesgue unit interval, as an agent space, can be extended to allow a rich Fubini extension; see Sun and Zhang (2009), and more generally Podczeck (2010). Also note that a rich Fubini extension is called a rich product probability space in Sun (2006).

[13] Given that $(I, \mathcal{I}, \lambda)$ is an atomless (complete) probability space, a single point (and thus up to countably many points) has a measure zero, and thus, essential pairwise independence is more general than the usual pairwise and mutual independence.

The pair $(\boldsymbol{g}, \boldsymbol{p})$ above is said to be a *mixed Walrasian equilibrium* (mixed WE) of the economy $\mathscr{E}$. Furthermore, we say that a mixed allocation $\boldsymbol{g}$ of $\mathscr{E}$ has the *ex post Walrasian property* under $\boldsymbol{p}$ if for $\mathbf{P}$-almost all $\omega \in \Omega$, $(\boldsymbol{g}_\omega, \boldsymbol{p})$ is a Walrasian equilibrium of $\mathscr{E}$.

Condition 1 above requires that almost every agent makes a (mixed) consumption plan to maximize the expected utility subject to the budget constraint. Condition 2 is the standard market clearing condition.

The next result addresses the relationship among Walrasian equilibrium, randomized Walrasian equilibrium, and mixed Walrasian equilibrium.

Proposition 1. *In a large economy $\mathscr{E}: I \to \mathcal{U}_{mo} \times \mathbb{R}^n_+$, we have the following statements:*

(i) For every mixed Walrasian equilibrium $(\boldsymbol{g}, \boldsymbol{p})$, if $\boldsymbol{h}$ is such that $\boldsymbol{h}(i) = \mathbf{P}(\boldsymbol{g}_i)^{-1}$ for all $i \in I$, then $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium. For every randomized Walrasian equilibrium $(\boldsymbol{h}, \boldsymbol{p})$, there exists a mixed Walrasian allocation $\boldsymbol{g}$ under $\boldsymbol{p}$ such that $\boldsymbol{h}(i) = \mathbf{P}(\boldsymbol{g}_i)^{-1}$ for all $i \in I$;

(ii) A mixed allocation is a mixed Walrasian allocation under $\boldsymbol{p}$ if and only if it has the ex post Walrasian property under $\boldsymbol{p}$.

Part (i) of the result above establishes that to model the strategic uncertainty, a randomized Walrasian equilibrium is equivalent to a mixed Walrasian equilibrium. Part (ii) suggests that after the resolution of uncertainty in a mixed Walrasian allocation, the realized consumption profile is a Walrasian allocation. Thus, the individual agents who choose optimized consumption bundles ex ante are ex post regret-free in the sense that their realized consumptions are still optimized. Namely, a mixed allocation $\boldsymbol{g}$ is a Walrasian allocation under $\boldsymbol{p}$ if any only if for almost all $\omega \in \Omega$, the realized allocation g_ω that is specified by the mixed allocation is a Walrasian allocation under the same $\boldsymbol{p}$.[14]

It is also worth noting that when the agent space is saturated,[15] it is a consequence of the saturation property to show that any Walrasian equilibrium distribution can be induced by a Walrasian equilibrium. It follows from Theorem 4.2 in Sun (2006) that the agent space in the framework of a rich Fubini extension will automatically have the saturation property. Thus, we can now delineate the relations of different equilibrium notions in a large economy with its agent space allowing a rich Fubini extension in Fig. 1.

4.2. Equilibrium notions in a large generalized game

We now consider a large generalized game that allows different players to have different action sets.[16] Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space representing the space of players and let

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[14] For a large economy with idiosyncratic shocks on the preferences and endowments, Theorem 3 in Sun (1999) showed the existence of an independent allocation process with the ex post Walrasian property. Since only deterministic economies are considered here, our concern is not the existence issue, but a general characterization of the ex post Walrasian property via the mixed Walrasian notion.

[15] An atomless probability space $(I, \mathcal{I}, \lambda)$ is said to be saturated, or have the saturation property if the measure λ restricted to a set of positive measure is never countably generated modulo the null sets; for details on the saturation property and its equivalent versions, see Keisler and Sun (2009).

[16] The authors thank an associate editor for suggesting us to work with an action correspondence to allow heterogeneity on players' action sets.

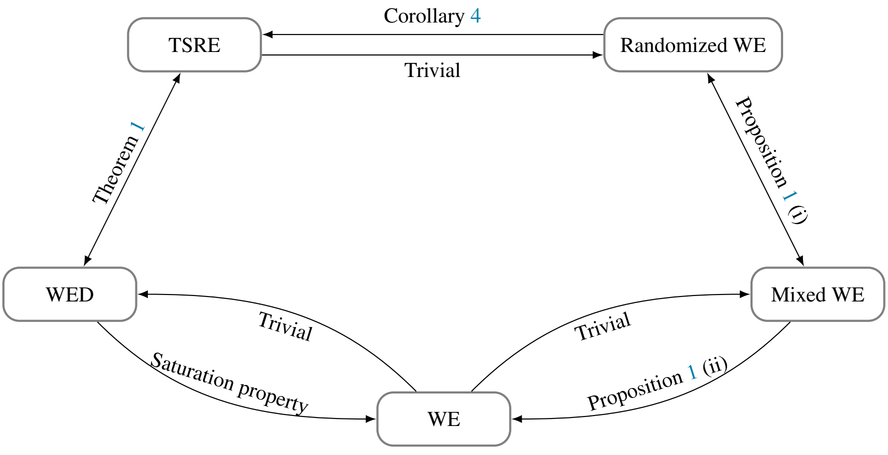

Fig. 1. Unification of equilibrium notions in a large economy.

A be a compact metric space representing a common action space. Let $\mathcal{A}: I \twoheadrightarrow A$ be a measurable correspondence with nonempty and compact values, which specifies the feasible action set $A_i = \mathcal{A}(i)$ for every $i \in I$. Let $\mathcal{D}_{\mathcal{A}}$ be the closure of $\{\lambda f^{-1} \mid f$ is a measurable selection of $\mathcal{A}\}$ in $\mathcal{M}(A)$ with the weak topology, representing the set of possible externalities or societal summaries. Let $\mathcal{C}_A$ be the set of all compact subsets of A, which is a compact metric space with the Hausdorff metric (and the resulting Borel σ-algebra); see Theorem 3.85 in Aliprantis and Border (2006) for example. Let $\mathcal{U}_{\mathcal{A}}$ be the space of all real-valued continuous functions on $A \times \mathcal{D}_{\mathcal{A}}$, representing the space of payoff functions, metrized by the supremum norm. The characteristics of each individual player consists of a feasible action set (i.e., a compact subset of A) and a payoff function, and thus the space of all possible characteristics is $\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$.

A *large generalized game* (that allows heterogeneous action sets) is a measurable function $\mathcal{G}^H (\equiv (\mathcal{A}, u))$ from $(I, \mathcal{I}, \lambda)$ to $\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$. Here, we slightly abuse the notation $\mathcal{A}$ by viewing it as the function $i \mapsto A_i$, which assigns the feasible action set for each player. By Aliprantis and Border (2006, Theorem 18.10), the action correspondence $\mathcal{A}: I \twoheadrightarrow A$ is measurable if and only if the function $i \mapsto A_i$ from I to $\mathcal{C}_A$ is Borel measurable.

Note that when the action correspondence is a constant correspondence $i \mapsto A$, a large generalized game reduces to a conventional large game[17] in Section 2.2. Also, note that an agent's characteristics in a large (abstract) economy contain two components: a preference as well as an endowment (which induces feasible consumption bundles). Thus, the formulation above brings more similarities between large games and large economies.

We are ready to elaborate those solution concepts of a conventional large game in Section 2.2 in a large generalized game.

[17] A large game with traits as considered in Khan et al. (2013) and Qiao and Yu (2014) may also be viewed as a special case of large generalized games considered in this section. To see this, recall that a large game with traits is a measurable function $\mathcal{G}^T$ from $(I, \mathcal{I}, \lambda)$ to $T \times \mathcal{U}_{(A',T)}$ where T is a compact metric space that represents the space of traits, and $\mathcal{U}_{(A',T)}$ is the space of all continuous payoff functions on a common compact action space A' as well as the space of all distributions on $T \times A'$. Denote the projections of $\mathcal{G}^T$ on T and $\mathcal{U}_{(A',T)}$ as α and u' respectively. Next, let $A = T \times A'$, $\mathcal{A} = (\alpha, \mathcal{A}')$ where $\mathcal{A}'$ is the constant correspondence on I taking the value A'. The large generalized game specified by $(\mathcal{A}, u')$ is now well-defined and is equivalent to the given large game with traits $\mathcal{G}^T$.

Definition 8 *(Pure-strategy Nash equilibrium).* A pure strategy profile f of a large generalized game $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$ is a measurable function from I to A such that for λ-almost all $i \in I$, $f(i) \in A_i$, and is said to be a *pure-strategy Nash equilibrium* (pure NE) if, for λ-almost all $i \in I$,

$$u_i\big(f(i), \lambda f^{-1}\big) \geq u_i\big(a, \lambda f^{-1}\big) \text{ for all } a \in A_i.$$

Definition 9 *(Nash equilibrium distribution).* A Borel probability measure τ on $\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \times A$ is said to be a *Nash equilibrium distribution* (NED) of a large generalized game in its distributional form $\lambda(\mathcal{G}^H)^{-1}$ if $\tau_{\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}} = \lambda(\mathcal{G}^H)^{-1}$, $\tau_A \in \mathcal{D}_{\mathcal{A}}$, and $\tau\big(\mathrm{Br}(\tau)\big) = 1$ where

$$\mathrm{Br}(\tau) = \big\{(A', v, a) \in \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \times A \mid a \in A', v(a, \tau_A) \geq v(x, \tau_A) \text{ for all } x \in A'\big\}.$$

A Nash equilibrium distribution τ of $\lambda(\mathcal{G}^H)^{-1}$ is *symmetric* if there exists a measurable function $s : \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \to A$ such that $\tau(\text{graph of } s) = 1$.

Definition 10 *(TSRE in large generalized games).* A randomized strategy profile h of a large generalized game $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$ is a measurable function from I to $\mathcal{M}(A)$ such that for λ-almost all $i \in I$, $h(i; A_i) = 1$, and is said to be a *randomized-strategy Nash equilibrium* (randomized NE) if, for λ-almost all $i \in I$,

$$\int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right) h(i; \mathrm{d}a) \geq \int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right) \mathrm{d}\eta(a)$$

for all $\eta \in \mathcal{M}(A_i)$. A randomized-strategy Nash equilibrium h of $\mathcal{G}^H$ is said to be a *type-symmetric randomized equilibrium* (TSRE) if $h(i) = h(j)$ whenever $\mathcal{G}^H(i) = \mathcal{G}^H(j)$.

We next generalize the corresponding results in Section 2.2 as follows.

Theorem 3. *Theorem 2, Corollary 1 and Corollary 2 still hold if we replace "a large game $\mathcal{G} : I \to \mathcal{U}_A$" by "a large generalized game $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$".*

We can also consider the notion of mixed-strategy Nash equilibrium.

Definition 11 *(Mixed-strategy Nash equilibrium).* A mixed strategy profile of a large generalized game $\mathcal{G}^H$ is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function $g : I \times \Omega \to A$ such that g is essentially pairwise independent, and for λ-almost all $i \in I$ and $\mathbf{P}$-almost all $\omega \in \Omega$, $g(i, \omega) \in A_i$. A mixed strategy profile g is said to be a *mixed-strategy Nash equilibrium* (mixed NE) of $\mathcal{G}^H$ if, for λ-almost all $i \in I$,

$$\int_\Omega u_i\big(g_i(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega) \tag{4}$$

for any random variable $\eta : \Omega \to A_i$. Furthermore, we say that a mixed strategy profile g of $\mathcal{G}^H$ has the *ex post Nash property* if, for $\mathbf{P}$-almost all $\omega \in \Omega$, g_ω is a pure-strategy Nash equilibrium of $\mathcal{G}^H$.

In Definition 11, Equation (4) requires that almost every agent's equilibrium strategy maximizes her expected payoff that takes account of others' mixed strategies. The next result addresses the relationship among pure-strategy, mixed-strategy and randomized-strategy Nash equilibria.

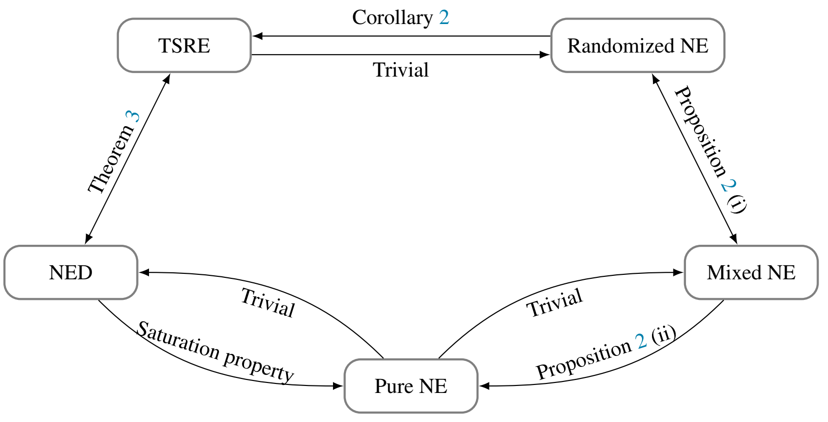

Fig. 2. Unification of equilibrium notions in a large generalized game.

Proposition 2. *In a large generalized game $\mathcal{G}^H: I \to \mathcal{C}_A \times \mathcal{U}_A$, we have the following statements:*

(i) For every mixed-strategy Nash equilibrium g, if h is such that $h(i) = \mathbf{P}(g_i)^{-1}$ for all $i \in I$, then h is a randomized-strategy Nash equilibrium. For every randomized-strategy Nash equilibrium h, there exists a mixed-strategy Nash equilibrium g such that $h(i) = \mathbf{P}(g_i)^{-1}$ for all $i \in I$;

(ii) A mixed strategy profile is a mixed-strategy Nash equilibrium if and only if it has the ex post Nash property.

We can now use Fig. 2 to conclude how various equilibrium notions would be unified for a large generalized game in the framework of a rich Fubini extension.

5. Proof of results

Lemma 1 deals with the properties of analytic sets used in the proofs of our theorems. For the definition of analytic sets and Lemma 1 (i), see Aliprantis and Border (2006, p. 446) and for Lemma 1 (ii), see Theorem 12.28 in Aliprantis and Border (2006).

Lemma 1. *(i) The continuous image in a Polish space of an analytic set is analytic. (ii) A function between Polish spaces is Borel measurable if and only if its graph is analytic.*

For the convenience of the reader, here we state the definition of a Fubini extension as in Sun (2006). Let probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be the index and sample spaces, respectively. Let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ be the usual product probability space. Given a function f on $I \times \Omega$ (not necessarily $\mathcal{I} \otimes \mathcal{F}$-measurable), for any $(i, \omega) \in I \times \Omega$, let f_i be the function $f(i, \cdot)$ on Ω and f_ω the function $f(\cdot, \omega)$ on I. A formal definition of the Fubini extension is given below.

Definition 12 *(Fubini extension).* A probability space $(I \times \Omega, \mathcal{W}, \mathbf{Q})$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ is said to be a *Fubini extension* of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ if for any real-valued $\mathbf{Q}$-integrable function f on $(I \times \Omega, \mathcal{W})$,

1. the two functions f_i and f_ω are integrable, respectively, on $(\Omega, \mathcal{F}, \mathbf{P})$ for λ-almost all $i \in I$, and on $(I, \mathcal{I}, \lambda)$ for $\mathbf{P}$-almost all $\omega \in \Omega$;
2. $\int_\Omega f_i \, d\mathbf{P}$ and $\int_I f_\omega \, d\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$, with $\int_{I\times\Omega} f \, d\mathbf{Q} = \int_I \big(\int_\Omega f_i \, d\mathbf{P}\big) \, d\lambda = \int_\Omega \big(\int_I f_\omega \, d\lambda\big) \, d\mathbf{P}$.

To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, \mathbf{Q})$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$.

A Fubini extension is *rich* if there is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process F from $I \times \Omega$ to $[0, 1]$ such that the random variables $F_i(\cdot) = F(i, \cdot)$ are independent and have the uniform distribution on $[0, 1]$.

We need two Lemmas related to the rich Fubini extension to prove our propositions. The first, which is on the universality property of a rich Fubini extension, is taken from Proposition 5.3 in Sun (2006). It says that one can construct processes on a rich Fubini extension with essentially pairwise independent random variables that take any given variety of distributions.

Lemma 2. *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ be a rich Fubini extension, X be a Polish space, and f a measurable mapping from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. Then, there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F\colon I \times \Omega \to X$ such that the process F is essentially pairwise independent and $f(i)$ is the induced distribution by F_i, for λ-almost all $i \in I$.*

We also need a version of the exact law of large games (ELLN) in the framework of the Fubini extension. The following lemma is taken from Corollary 2.9 of Sun (2006).

Lemma 3. *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ be a rich Fubini extension and X be a Polish space. If $F\colon I \times \Omega \to X$ is an essentially pairwise independent and an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process, then the sample distribution λF_ω^{-1} is the same as the distribution $(\lambda \boxtimes \mathbf{P}) F^{-1}$ for $\mathbf{P}$-almost all $\omega \in \Omega$.*

5.1. Proof of Theorem 1

Lemma 4. *Let $\boldsymbol{h}$ be a randomized allocation of a large economy $\mathscr{E}\colon I \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$. Then, $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium of $\mathscr{E}$ if and only if the type-consumption distribution induced by $\boldsymbol{h}$ is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$.*

Proof. Let $\tau = \int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j) \, d\lambda(j)$, the type-consumption distribution induced by the randomized allocation $\boldsymbol{h}$ of $\mathscr{E}$. It is clear that

$$\int_I \delta_{\succsim(j)} \otimes \boldsymbol{h}(j) \, d\lambda(j) = \tau_{\mathscr{P}_{mo}\times\mathbb{R}^{n,2}_+} \text{ and } \lambda\mathscr{E}^{-1} = \tau_{\mathscr{P}_{mo}\times\mathbb{R}^{n,1}_+}.$$

Suppose that $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium of $\mathscr{E}$. Then, it follows from Condition 2 in Definition 5 that

$$\int_I \int_{\mathbb{R}^n_+} \boldsymbol{x}\boldsymbol{h}(i; d\boldsymbol{x}) \, d\lambda(i) = \int_I \boldsymbol{e}(i) \, d\lambda(i),$$

which implies

$$\int_{\mathbb{R}_+^n} \boldsymbol{x} \,\mathrm{d}\Big(\int_I \boldsymbol{h}(i)\,\mathrm{d}\lambda(i)\Big) = \int_I \boldsymbol{e}(i)\,\mathrm{d}\lambda(i).$$

By the change of variables theorem, the above equation is equivalent to

$$\int_{\mathbb{R}_+^{n,2}} \boldsymbol{x}\,\mathrm{d}\nu = \int_{\mathscr{P}_{mo}\times\mathbb{R}_+^{n,1}} \boldsymbol{e}\,\mathrm{d}\lambda\mathscr{E}^{-1},$$

where ν is the marginal distribution of τ on the commodity space. We now have verified Conditions 1 and 3 in Definition 2. Next, Condition 1 in Definition 5 implies that for λ-almost all $i \in I$, $\boldsymbol{h}(i) \in \mathcal{M}(\mathbb{R}_+^{n,2})$ has its support in $D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$. Condition 2 in Definition 2 now follows as $\tau(E_{\boldsymbol{p}}) = \int_I \big(\delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\big)(E_{\boldsymbol{p}})\,\mathrm{d}\lambda(j) = 1$. Therefore, τ is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$.

Now, suppose that $\tau = \int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\,\mathrm{d}\lambda(j)$ is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$. Then, we have that $\int_I \big(\delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\big)(E_{\boldsymbol{p}})\,\mathrm{d}\lambda(j) = \tau(E_{\boldsymbol{p}}) = 1$. Thus, for λ-almost all $i \in I$, $\big(\delta_{\mathscr{E}(i)} \otimes \boldsymbol{h}(i)\big)(E_{\boldsymbol{p}}) = 1$. Therefore, for λ-almost all $i \in I$,

$$\begin{aligned}\boldsymbol{h}(i)\big(D(\boldsymbol{p}, \succsim_i, \boldsymbol{e}_i)\big) &= \big(\delta_{\mathscr{E}(i)} \otimes \boldsymbol{h}(i)\big)\big(\{(\succsim, \boldsymbol{e}, \boldsymbol{x}) \mid \boldsymbol{x} \in D(\boldsymbol{p}, \succsim, \boldsymbol{e})\}\big)\\ &= \big(\delta_{\mathscr{E}(i)} \otimes \boldsymbol{h}(i)\big)(E_{\boldsymbol{p}}) = 1.\end{aligned}$$

That is, for λ-almost all $i \in I$, $\boldsymbol{h}(i)$ has the support in $D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$. Since τ is a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, we have that

$$\int_I \boldsymbol{e}(i)\,\mathrm{d}\lambda(i) = \int_{\mathscr{P}_{mo}\times\mathbb{R}_+^{n,1}} \boldsymbol{e}\,\mathrm{d}\lambda\mathscr{E}^{-1} = \int_{\mathbb{R}_+^{n,2}} \boldsymbol{x}\,\mathrm{d}\Big(\int_I \boldsymbol{h}(j)\,\mathrm{d}\lambda(j)\Big) = \int_I \int_{\mathbb{R}_+^n} \boldsymbol{x}\boldsymbol{h}(i; \mathrm{d}\boldsymbol{x})\,\mathrm{d}\lambda(i).$$

This completes the proof. □

Proof of Theorem 1. Since any TSRE of a large economy is a randomized Walrasian equilibrium of the economy, Lemma 4 implies (i) directly.

We next show that (ii) every Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ can be "lifted" to some TSRE in $\mathscr{E}$. Let τ be a Walrasian equilibrium distribution of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, and let $\kappa\colon \mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \to \mathcal{M}(\mathbb{R}_+^{n,2})$ be the disintegration[18] of τ with respect to $\lambda\mathscr{E}^{-1}$. Let $\boldsymbol{h} = \kappa \circ \mathscr{E}\colon I \to \mathcal{M}(\mathbb{R}_+^{n,2})$. By the change of variables theorem and the property of the disintegration, we have that

$$\int_I \delta_{\mathscr{E}(j)} \otimes \boldsymbol{h}(j)\,\mathrm{d}\lambda(j) = \int_{\mathscr{P}_{mo}\times\mathbb{R}_+^{n,1}} \delta_{(\succsim,\boldsymbol{e})} \otimes \kappa(\succsim, \boldsymbol{e})\,\mathrm{d}\lambda\mathscr{E}^{-1}(\succsim, \boldsymbol{e}) = \tau.$$

Thus, τ is the type-consumption distribution induced by $\boldsymbol{h}$. In addition, from the construction of h, it is clear that $h(i) = h(j)$ whenever $\mathscr{E}(i) = \mathscr{E}(j)$ for any $i, j \in I$. Together with Lemma 4, we can conclude that $(\boldsymbol{h}, \boldsymbol{p})$ is a TSRE of $\mathscr{E}$.

[18] The disintegration of a probability measure τ on two polish spaces with respect to a given marginal $\lambda\mathscr{E}^{-1}$ on one polish space always exist and is unique ($\lambda\mathscr{E}^{-1}$-almost everywhere); see Proposition 3.6 in Crauel (2002) for example. Notice that the disintegration in the case of large games with non-metrizable action sets has also been considered in Khan (1989). Note that a disintegration is also known as a regular conditional distribution in probability theory; see Dudley (2002, pp. 342–345) for related discussion.

We now prove statement (iii). Let $\mathscr{E}^l$ be a large economy with the Lebesgue agent space $([0,1], \mathcal{B}[0,1], \ell)$. Let τ be a Walrasian equilibrium distribution of $\ell(\mathscr{E}^l)^{-1}$ under $\boldsymbol{p}$ and let $\kappa^l\colon \mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \to \mathcal{M}(\mathbb{R}_+^{n,2})$ be the disintegration of τ with respect to $\ell(\mathscr{E}^l)^{-1}$. From the proof of statement (ii) above, we know that $(\boldsymbol{h}^l := \kappa^l \circ \mathscr{E}^l, \boldsymbol{p})$ is a TSRE of $\mathscr{E}^l$. Let $(\boldsymbol{h}', \boldsymbol{p})$ be another TSRE of $\mathscr{E}^l$ that induces τ. To complete the proof, it is enough to show that for ℓ-almost all $i \in [0,1]$, $\boldsymbol{h}'(i) = \boldsymbol{h}^l(i)$.

Consider the mapping

$$(\mathscr{E}^l, \boldsymbol{h}')\colon [0,1] \to \mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \times \mathcal{M}(\mathbb{R}_+^{n,2}).$$

Since both $\mathscr{E}^l$ and $\boldsymbol{h}'$ are Borel measurable, so is $(\mathscr{E}^l, \boldsymbol{h}')$. Thus, Lemma 1 (ii) implies that $\mathrm{Graph}(\mathscr{E}^l, \boldsymbol{h}')$ is analytic. Fix $\boldsymbol{x}_0 \in \mathbb{R}_+^{n,2}$ and let $H\colon \mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \to \mathcal{M}(\mathbb{R}_+^{n,2})$ be a mapping such that

$$H(\succsim, \boldsymbol{e}) = \begin{cases} \boldsymbol{h}'(i), & \text{if } (\succsim, \boldsymbol{e}) = \mathscr{E}^l(i); \\ \delta_{\boldsymbol{x}_0}, & \text{otherwise.} \end{cases}$$

Let $H|_{\mathrm{range}(\mathscr{E}^l)}\colon \mathrm{range}(\mathscr{E}^l) \to \mathcal{M}(\mathbb{R}_+^{n,2})$ be the restriction of H on the range of $\mathscr{E}^l$. From the construction, we have

$$\mathrm{Graph}\big(H|_{\mathrm{range}(\mathscr{E}^l)}\big) = \mathrm{Proj}_{\mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \times \mathcal{M}(\mathbb{R}_+^{n,2})} \mathrm{Graph}(\mathscr{E}^l, \boldsymbol{h}').$$

Since the projection mapping is continuous and $\mathrm{Graph}(\mathscr{E}^l, \boldsymbol{h}')$ is analytic, we can now appeal to Lemma 1 (i) to assert that $\mathrm{Graph}\big(H|_{\mathrm{range}(\mathscr{E}^l)}\big)$ is also analytic. Thus, by Lemma 1 (ii) again, we can say that $H|_{\mathrm{range}(\mathscr{E}^l)}$ is Borel measurable. Then, there exists a Borel measurable mapping $H'\colon \mathscr{P}_{mo} \times \mathbb{R}_+^{n,1} \to \mathcal{M}(\mathbb{R}_+^{n,2})$ such that $H|_{\mathrm{range}(\mathscr{E}^l)}$ is also the restriction of H' on $\mathrm{range}(\mathscr{E}^l)$; see, for example, Theorem 4.2.5 in Dudley (2002) and the discussion below it.[19] Furthermore, from the construction, it is clear that $\boldsymbol{h}' = H \circ \mathscr{E}^l = H|_{\mathrm{range}(\mathscr{E}^l)} \circ \mathscr{E}^l = H' \circ \mathscr{E}^l$. Since $\boldsymbol{h}'$ is a TSRE allocation under $\boldsymbol{p}$ that induces τ, we now have

$$\tau = \int_I \delta_{\mathscr{E}^l(j)} \otimes \boldsymbol{h}'(j)\, \mathrm{d}\ell(j) = \int_{\mathscr{P}_{mo} \times \mathbb{R}_+^{n,1}} \delta_{(\succsim, \boldsymbol{e})} \otimes H'(\succsim, \boldsymbol{e})\, \mathrm{d}\ell(\mathscr{E}^l)^{-1}.$$

Thus, H' is also a disintegration of τ with respect to $\ell(\mathscr{E}^l)^{-1}$. By the ($\ell(\mathscr{E}^l)^{-1}$-almost everywhere) uniqueness of the disintegration, H' and k^l are $\ell(\mathscr{E}^l)^{-1}$-almost everywhere the same. Therefore, $\boldsymbol{h}'(= H' \circ \mathscr{E}^l)$ and $\boldsymbol{h}^l(= k^l \circ \mathscr{E}^l)$ must also be ℓ-almost everywhere the same. It is now clear that (iii) must hold. □

5.2. Proof of Theorem 3

Note that as a conventional large game $\mathcal{G}\colon I \to \mathcal{U}_A$ in Section 2.2 can be viewed as a special case of a large generalized game $\mathcal{G}^H\colon I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$ in Section 4, it is clear that Theorem 3 implies Theorem 2, Corollary 1, and Corollary 2. Thus, we will only prove Theorem 3. We first prove the following lemma.

[19] The range space $\mathbb{R}$ in (Dudley, 2002, Theorem 4.2.5) can be replaced by any Polish space with its Borel σ-algebra due to the following reason: between any two uncountable Polish spaces, there is a Borel isomorphism—a bijection that preserves the Borel structures.

Lemma 5. *If h is a randomized strategy profile of a large generalized game $\mathcal{G}^H: I \to \mathcal{C}_A \times \mathcal{U}_\mathcal{A}$, then h is a randomized-strategy Nash equilibrium of $\mathcal{G}^H$ if and only if the type-action distribution induced by h is a Nash equilibrium distribution of $\lambda(\mathcal{G}^H)^{-1}$.*

Proof. Let $\tau = \int_I \delta_{\mathcal{G}^H(j)} \otimes h(j)\,\mathrm{d}\lambda(j)$, the type-action distribution induced by h of $\mathcal{G}^H$. It is clear that

$$\tau_{\mathcal{C}_A \times \mathcal{U}_\mathcal{A}} = \lambda(\mathcal{G}^H)^{-1} \text{ and } \tau_A = \int_I h(j)\,\mathrm{d}\lambda(j) \in \mathcal{D}_\mathcal{A}.$$

By definition, if h is a randomized-strategy Nash equilibrium of $\mathcal{G}^H$, then for λ-almost all $i \in I$, we have

$$\int_{A_i} u_i\Big(a, \int_I h(j)\,\mathrm{d}\lambda(j)\Big) h(i; \mathrm{d}a) \geq \int_{A_i} u_i\Big(a, \int_I h(j)\,\mathrm{d}\lambda(j)\Big) \eta(\mathrm{d}a)$$

for all $\eta \in \mathcal{M}(A_i)$. This is equivalent to that for λ-almost all $i \in I$, we have

$$\int_{A_i} u_i(a, \tau_A) h(i; \mathrm{d}a) \geq \int_{A_i} u_i(a, \tau_A)\eta(\mathrm{d}a)$$

for all $\eta \in \mathcal{M}(A_i)$. The above inequality holds if and only if for λ-almost all $i \in I$, h_i-almost all $a \in A_i$,

$$u_i(a, \tau_A) = \max_{a' \in A_i} u_i(a', \tau_A).$$

The above equation holds if and only if for λ-almost all $i \in I$,

$$\begin{aligned} 1 = h(i)\big(\arg\max_{x \in A_i} u_i(x, \tau_A)\big) &= \big(\delta_{\mathcal{G}^H(i)} \otimes h(i)\big)\big(\{(\mathcal{G}^H(i), a) \mid a \in \arg\max_{x \in A_i} u_i(x, \tau_A)\}\big) \\ &= \big(\delta_{\mathcal{G}^H(i)} \otimes h(i)\big)\big(\mathrm{Br}(\tau)\big), \end{aligned}$$

which is equivalent to

$$1 = \int_I \delta_{\mathcal{G}^H(j)} \otimes h(j)\,\mathrm{d}\lambda(j)(\mathrm{Br}(\tau)\big) = \tau\big(\mathrm{Br}(\tau)\big).$$

The proof is now complete. $\square$

Next, we will first prove a generalization of Theorem 2 by allowing $\mathcal{G}: I \to \mathcal{U}_A$ in the statements of Theorem 2 to be a large generalized game $\mathcal{G}^H: I \to \mathcal{C}_A \times \mathcal{U}_\mathcal{A}$.

Proof. As any TSRE of a large generalized game is a randomized-strategy Nash equilibrium of the game, Lemma 5 implies (i).

We next show that (ii) every Nash equilibrium distribution of $\lambda(\mathcal{G}^H)^{-1}$ can be "lifted" to some TSRE in $\mathcal{G}^H$. Let τ be a Nash equilibrium distribution of $\lambda(\mathcal{G}^H)^{-1}$ and $k: \mathcal{C}_A \times \mathcal{U}_\mathcal{A} \to \mathcal{M}(A)$ the disintegration of τ with respect to $\lambda(\mathcal{G}^H)^{-1}$. Let $h = k \circ \mathcal{G}^H: I \to \mathcal{M}(A)$. It is clear that $h(i) = h(j)$ whenever $\mathcal{G}^H(i) = \mathcal{G}^H(j)$ for any $i, j \in I$. Thus, to show that h is a TSRE, we only need to show that h is a randomized-strategy Nash equilibrium.

Towards this end, by the change of variables theorem and the property of the disintegration, we have that

$$\int_I \delta_{\mathcal{G}^H(j)} \otimes h(j)\,\mathrm{d}\lambda(j) = \int_{\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}} \delta_{(A',v)} \otimes k(A',v)\,\mathrm{d}\lambda(\mathcal{G}^H)^{-1}(A',v) = \tau.$$

Furthermore, since $\tau\big(\mathrm{Br}(\tau)\big) = 1$, we have that for λ-almost all $j \in I$,

$$1 = \big(\delta_{\mathcal{G}^H(j)} \otimes h(j)\big)\big(\mathrm{Br}(\tau)\big) = \big(\delta_{(A_j,u_j)} \otimes h(j)\big)\big(\{(A_j,u_j,a) \mid a \in \arg\max_{x \in A_j} u_j(x,\tau_A)\}\big),$$

which implies

$$h(j)\big(\arg\max_{x \in A_j} u_j(x,\tau_A)\big) = 1,$$

and hence $h(j)$ has the support in A_j. Therefore, h is a randomized strategy profile of $\mathcal{G}^H$. By Lemma 5, h is a randomized-strategy Nash equilibrium of $\mathcal{G}^H$. We have now shown that h is a TSRE of $\mathcal{G}^H$ that induces the given τ.

We only need to prove statement (iii). Let $\mathcal{G}^l$ be a large generalized game with the Lebesgue player space $([0,1], \mathcal{B}[0,1], \ell)$. Let τ be a Nash equilibrium distribution of $\ell(\mathcal{G}^l)^{-1}$ and let $k^l : \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \to \mathcal{M}(A)$ be the disintegration of τ with respect to $\ell(\mathcal{G}^l)^{-1}$. From the proof of statement (ii) above, we know that $h^l := k^l \circ \mathcal{G}^l$ is a TSRE of $\mathcal{G}^l$. We next show that it is also the unique TSRE.

Towards this end, let h' be another TSRE of $\mathcal{G}^l$ that induces τ. It is enough to show that for ℓ-almost all $i \in I$, $h'(i) = h^l(i)$. Consider the mapping

$$(\mathcal{G}^l, h') : [0,1] \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \times \mathcal{M}(A).$$

Since both $\mathcal{G}^l$ and h' are Borel measurable, so is $(\mathcal{G}^l, h')$. By Lemma 1 (ii), Graph$(\mathcal{G}^l, h')$ is analytic.

Fix $a_0 \in A$ and let $H : \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \to \mathcal{M}(A)$ be a mapping such that

$$H(A',v) = \begin{cases} h'(i), & \text{if } (A',v) = \mathcal{G}^l(i); \\ \delta_{a_0}, & \text{otherwise.} \end{cases}$$

Let $H|_{\mathrm{range}(\mathcal{G}^l)} :$ range$(\mathcal{G}^l) \to \mathcal{M}(A)$ be the restriction of H on the range of $\mathcal{G}^l$. From the construction, the graph of $H|_{\mathrm{range}(\mathcal{G}^l)}$,

$$\mathrm{Graph}\big(H|_{\mathrm{range}(\mathcal{G}^l)}\big) = \mathrm{Proj}_{\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \times \mathcal{M}(A)}\,\mathrm{Graph}(\mathcal{G}^l, h'),$$

the projection of the graph of $(\mathcal{G}^l, h')$ on $\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \times \mathcal{M}(A)$. Since the projection mapping is continuous and Graph$(\mathcal{G}^l, h')$ is analytic, we can now appeal to Lemma 1 (i) to assert that Graph$\big(H|_{\mathrm{range}(\mathcal{G}^l)}\big)$ is also analytic. Thus, by Lemma 1 (ii) again, we can say that $H|_{\mathrm{range}(\mathcal{G}^l)}$ is Borel measurable. Then, there exists a Borel measurable mapping $H' : \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \to \mathcal{M}(A)$ such that $H|_{\mathrm{range}(\mathcal{G}^l)}$ is also the restriction of H' on range$(\mathcal{G}^l)$; see, for example, Theorem 4.2.5 in Dudley (2002) and Footnote 19. Furthermore, from the construction, it is clear that $h' = H \circ \mathcal{G}^l = H|_{\mathrm{range}(\mathcal{G}^l)} \circ \mathcal{G}^l = H' \circ \mathcal{G}^l$. Since h' is a TSRE that induces τ, we now have

$$\tau = \int_I \delta_{\mathcal{G}^l(j)} \otimes h'(j)\,\mathrm{d}\ell(j) = \int_{\mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}} \delta_{(A',v)} \otimes H'(A',v)\,\mathrm{d}\ell(\mathcal{G}^l)^{-1}(A',v).$$

This is to say, H' is also a disintegration of τ with respect to $\ell(\mathcal{G}^l)^{-1}$. By the ($\ell(\mathcal{G}^l)^{-1}$-almost everywhere) uniqueness of the disintegration, H' and k^l are $\ell(\mathcal{G}^l)^{-1}$-almost everywhere the same. So, $h'(= H' \circ \mathcal{G}^l)$ and $h^l(= k^l \circ \mathcal{G}^l)$ must also be ℓ-almost everywhere the same. This completes the proof. □

We now prove a generalization of Corollary 1 where $\mathcal{G}\colon I \to \mathcal{U}_A$ in Corollary 1 is replaced by $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$.

Proof. Let $\mathcal{G}^H$ be a large generalized game with the Lebesgue unit interval as its player space, and τ^s be a symmetric Nash equilibrium distribution of its distributional form $l(\mathcal{G}^H)^{-1}$. Then there is a measurable function $s\colon \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}} \to A$ such that $\tau^s(\text{graph of } s) = 1$. We can construct a measurable function $f\colon [0, 1] \to A$ so that $f = s \circ \mathcal{G}^H$. It is easy to check that f is a symmetric pure-strategy Nash equilibrium (hence a TSRE) of $\mathcal{G}^H$ such that $\tau^s = \ell(\mathcal{G}^H, f)^{-1}$. As τ^s is a Nash equilibrium distribution, the analog of Theorem 2 (iii) for $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$ implies that there is a unique TSRE that induces τ^s. Hence f is the unique symmetric pure-strategy Nash equilibrium that induces τ^s. □

We next complete the proof of Theorem 3 by proving the analog of Corollary 2 for a large generalized game $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$ as below.

Proof. Lemma 5 implies that the type-action distribution induced by a randomized-strategy Nash equilibrium is a Nash equilibrium distribution. Then by the analog of Theorem 2 (ii) for $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$, we can find a TSRE that induces the same Nash equilibrium distribution. □

5.3. *Proof of Proposition 1*

Proof. We first prove statement (i). Suppose that $(\boldsymbol{g}, \boldsymbol{p})$ is a mixed Walrasian equilibrium. For each $i \in I$, let $\boldsymbol{h}(i) = \mathbf{P}(\boldsymbol{g}_i)^{-1}$. By Lemma 3, for $\mathbf{P}$-almost all $\omega \in \Omega$, we have

$$\int_I \boldsymbol{h}(i)\, \mathrm{d}\lambda(i) = \int_I \mathbf{P}(\boldsymbol{g}_i)^{-1}\, \mathrm{d}\lambda(i) = \lambda(\boldsymbol{g}_\omega)^{-1}. \tag{5}$$

Since $(\boldsymbol{g}, \boldsymbol{p})$ is a mixed Walrasian equilibrium, for λ-almost all $i \in I$,

$$\int_\Omega u_i(\boldsymbol{g}_i(\omega))\, \mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i(\boldsymbol{g}'(\omega))\, \mathrm{d}\mathbf{P}(\omega)$$

for each random variable $\boldsymbol{g}'\colon \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{g}'(\omega) \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$ for $\mathbf{P}$-almost all $\omega \in \Omega$. As $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless, for any $\boldsymbol{h}' \in \mathcal{M}(\mathbb{R}^n_+)$ with its support in agent i's budget set, there exists a random variable $\boldsymbol{g}'\colon \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{g}'(\omega) \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$ for $\mathbf{P}$-almost all $\omega \in \Omega$ and $\mathbf{P}(\boldsymbol{g}')^{-1} = \boldsymbol{h}'$. By Equation (5), and the change of variables theorem, we have that for λ-almost all $i \in I$,

$$\int_{\mathbb{R}^n_+} u_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{h}_i(\boldsymbol{x}) \geq \int_{\mathbb{R}^n_+} u_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{h}'(\boldsymbol{x})$$

for each $\boldsymbol{h}' \in \mathcal{M}(\mathbb{R}^n_+)$ with its support in agent i's budget set. Thus, for λ-almost all $i \in I$, $\boldsymbol{h}(i)$ has its support in $D\big(\boldsymbol{p}, u_i, \boldsymbol{e}(i)\big)$. Meanwhile, by Equation (5), for $\mathbf{P}$-almost all $\omega \in \Omega$, we also have

$$\int_I \int_{\mathbb{R}^n_+} \boldsymbol{x} \boldsymbol{h}(i; \mathrm{d}\boldsymbol{x}) \, \mathrm{d}\lambda(i) = \int_{\mathbb{R}^n_+} \boldsymbol{x} \, \mathrm{d}\left(\int_I \boldsymbol{h}(i) \, \mathrm{d}\lambda(i) \right) = \int_{\mathbb{R}^n_+} \boldsymbol{x} \, \mathrm{d}\lambda \boldsymbol{g}_\omega^{-1}(\boldsymbol{x}) = \int_I \boldsymbol{g}_\omega(i) \, \mathrm{d}\lambda(i). \tag{6}$$

Since $(\boldsymbol{g}, \boldsymbol{p})$ is a mixed Walrasian equilibrium, we know that for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\int_I \boldsymbol{g}_\omega(i) \, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i) \, \mathrm{d}\lambda(i).$$

We have shown that $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium.

Next, suppose that $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium. Given that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ is a rich Fubini extension, by Lemma 2, there exists a mixed allocation $\boldsymbol{g}$ such that for λ-almost all $i \in I$,

$$\mathbf{P}(\boldsymbol{g}_i)^{-1} = \boldsymbol{h}(i).$$

By Lemma 3, we can also have Equations (5) and (6) hold for $\boldsymbol{h}$ and $\boldsymbol{g}$. As $(\boldsymbol{h}, \boldsymbol{p})$ is a randomized Walrasian equilibrium, we know that $\int_I \int_{\mathbb{R}^n_+} \boldsymbol{x} \boldsymbol{h}(i; \mathrm{d}\boldsymbol{x}) \, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i) \, \mathrm{d}\lambda(i)$. Together with Equation (6), we have

$$\int_I \boldsymbol{g}_\omega(i) \, \mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i) \, \mathrm{d}\lambda(i).$$

Meanwhile, we also know that $\boldsymbol{h}_i \in \mathcal{M}(\mathbb{R}^n_+)$ has its support in $D(\boldsymbol{p}, u_i, \boldsymbol{e}_i)$. Namely,

$$\int_{\mathbb{R}^n_+} u_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{h}_i(\boldsymbol{x}) \geq \int_{\mathbb{R}^n_+} u_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{h}'(\boldsymbol{x})$$

for every $\boldsymbol{h}' \in \mathcal{M}(\mathbb{R}^n_+)$ with support in agent i's budget set. Thus, by Equation (5) and the change of variables theorem, we have

$$\int_\Omega u_i\big(\boldsymbol{g}_i(\omega)\big) \, \mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\boldsymbol{g}'(\omega)\big) \, \mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\boldsymbol{g}' \colon \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{g}'(\omega) \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$. Thus, $(\boldsymbol{g}, \boldsymbol{p})$ is a mixed Walrasian equilibrium.

We now prove statement (ii). If a mixed allocation has the ex post Walrasian property under $\boldsymbol{p}$, it is easy to show that the mixed allocation is a mixed Walrasian allocation under $\boldsymbol{p}$. We only need to prove the "only if" part. Towards the end, suppose that $\boldsymbol{g}$ is a mixed Walrasian allocation under $\boldsymbol{p}$. Then, for λ-almost all $i \in I$,

$$\int_\Omega u_i\big(\boldsymbol{g}_i(\omega)\big) \, \mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\boldsymbol{g}'(\omega)\big) \, \mathrm{d}\mathbf{P}(\omega),$$

for every random variable $\boldsymbol{g}' \colon \Omega \to \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{g}'(\omega) \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$ for $\mathbf{P}$-almost all $\omega \in \Omega$. This implies that for λ-almost all $i \in I$ and $\mathbf{P}$-almost all $\omega \in \Omega$,

$$u_i\big(\boldsymbol{g}_i(\omega)\big) \geq u_i(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$. Then, by the Fubini property, we have, for $\mathbf{P}$-almost all $\omega \in \Omega$ and for λ-almost all $i \in I$,

$$u_i\big(\boldsymbol{g}_\omega(i)\big) \geq u_i(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \mathbb{R}^n_+$ such that $\boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{e}(i)$. That is, for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\boldsymbol{g}_\omega(i) \in D\big(\boldsymbol{p}, u_i, \boldsymbol{e}(i)\big).$$

Furthermore, for $\mathbf{P}$-almost all $\omega \in \Omega$, $\int_I \boldsymbol{g}_\omega(i)\,\mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i)\,\mathrm{d}\lambda(i)$. Therefore, for $\mathbf{P}$-almost all $\omega \in \Omega$, $(\boldsymbol{g}_\omega, \boldsymbol{p})$ is a Walrasian equilibrium. This completes the proof. □

5.4. *Proof of Proposition 2*

Proof. We first prove statement (i). Suppose that g is a mixed-strategy Nash equilibrium of $\mathcal{G}^H : I \to \mathcal{C}_A \times \mathcal{U}_{\mathcal{A}}$. Then, for λ-almost all $i \in I$, we have

$$\int_\Omega u_i\big(g_i(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\eta \colon \Omega \to A_i$. Let $h(i) = \mathbf{P} g_i^{-1}$ for all $i \in I$. Clearly, h is a randomized strategy profile of $\mathcal{G}^H$. By Lemma 3, it is clear that for $\mathbf{P}$-almost all $\omega \in \Omega$, $\lambda g_\omega^{-1} = \int_I \mathbf{P} g_j^{-1}\,\mathrm{d}\lambda(j) = \int_I h(j)\,\mathrm{d}\lambda(j)$. Hence, for λ-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i(\omega), \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\left(\eta(\omega), \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\mathbf{P}(\omega)$$

for any random variable $\eta \colon \Omega \to A_i$. Moreover, as $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless, for any $\xi \in \mathcal{M}(A_i)$, there exists a random variable $\eta \colon \Omega \to A_i$ such that $\xi = \mathbf{P}\eta^{-1}$. Therefore, we can again apply the change of variables theorem to assert that for λ-almost all $i \in I$,

$$\int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right) h(i; \mathrm{d}a) \geq \int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\xi(a)$$

for $\xi \in \mathcal{M}(A_i)$. Thus, h is a randomized-strategy Nash equilibrium.

Now suppose that h is a randomized-strategy Nash equilibrium of $\mathcal{G}^H$. Clearly, h is a measurable function from I to $\mathcal{M}(A)$ such that for λ-almost all $i \in I$, $h(i; A_i) = 1$. Thus, given that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ is a rich Fubini extension, by Lemma 2, there is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g from $I \times \Omega$ to A such that g is essentially pairwise independent and distribution $\mathbf{P} g_i^{-1}$ is the given distribution $h(i)$ for λ-almost all $i \in I$. Clearly, g is a mixed strategy profile of $\mathcal{G}^H$. Since h is a randomized-strategy Nash equilibrium, we have that for λ-almost all $i \in I$,

$$\int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right) h(i; \mathrm{d}a) \geq \int_{A_i} u_i\left(a, \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\xi(a)$$

for every $\xi \in \mathcal{M}(A_i)$. By the change of variables theorem, we have that for λ-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i(\omega), \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\left(\eta(\omega), \int_I h(j)\,\mathrm{d}\lambda(j)\right)\mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\eta\colon \Omega \to A_i$. Then we can apply Lemma 3 again to assert that for $\mathbf{P}$-almost all $\omega \in \Omega$, $\int_I h(j)\,\mathrm{d}\lambda(j) = \int_I \mathbf{P}g_j^{-1}\,\mathrm{d}\lambda(j) = \lambda g_\omega^{-1}$. Hence, λ-almost all $i \in I$,

$$\int_\Omega u_i\big(g_i(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \lambda g_\omega^{-1}\big)\,\mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\eta\colon \Omega \to A_i$. That is, g is a mixed-strategy Nash equilibrium.

We next prove statement (ii). Let g be a mixed strategy profile of $\mathcal{G}^H$. By Lemma 3, for $\mathbf{P}$-almost all $\omega \in \Omega$, $\lambda g_\omega^{-1} = \int_I \mathbf{P}g_j^{-1}\,\mathrm{d}\lambda(j)$. Let

$$\xi = \int_I \mathbf{P}g_j^{-1}\,\mathrm{d}\lambda(j).$$

Suppose that g is a mixed-strategy Nash equilibrium. Then we have that for λ-almost all $i \in I$,

$$\int_\Omega u_i\big(g_i(\omega), \xi\big)\,\mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \xi\big)\,\mathrm{d}\mathbf{P}(\omega)$$

for any random variable $\eta\colon \Omega \to A_i$. It implies that for λ-almost $i \in I$ and for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$u_i\big(g_i(\omega), \xi\big) = \max_{a \in A_i} u_i(a, \xi).$$

Then, by the Fubini property of a Fubini extension, we have, for $\mathbf{P}$-almost all $\omega \in \Omega$ and for λ-almost all $i \in I$,

$$u_i\big(g_\omega(i), \xi\big) = \max_{a \in A_i} u_i(a, \xi).$$

Thus, for $\mathbf{P}$-almost all $\omega \in \Omega$ and for λ-almost all $i \in I$,

$$u_i\big(g_\omega(i), \lambda g_\omega^{-1}\big) = \max_{a \in A_i} u_i\big(a, \lambda g_\omega^{-1}\big).$$

This means, for $\mathbf{P}$-almost all $\omega \in \Omega$, g_ω is a pure-strategy Nash equilibrium and, therefore, g has the ex post Nash property.

Now suppose that a mixed strategy profile g has the ex post Nash property in $\mathcal{G}^H$. Hence, for $\mathbf{P}$-almost all $\omega \in \Omega$ and for λ-almost all $i \in I$,

$$u_i\big(g_\omega(i), \lambda g_\omega^{-1}\big) = \max_{a \in A_i} u_i\big(a, \lambda g_\omega^{-1}\big).$$

Based on the Fubini property of a Fubini extension, we have, for λ-almost all $i \in I$ and for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$u_i\big(g_i(\omega), \lambda g_\omega^{-1}\big) = \max_{a \in A_i} u_i\big(a, \lambda g_\omega^{-1}\big).$$

Then, for λ-almost all $i \in I$ and for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$u_i\big(g_i(\omega), \xi\big) = \max_{a \in A_i} u_i(a, \xi).$$

Thus, for any random variable $\eta\colon \Omega \to A_i$, we have, for λ-almost all $i \in I$ and for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$u_i\big(g_i(\omega), \xi\big) \geq u_i\big(\eta(\omega), \xi\big),$$

which implies that for λ-almost all $i \in I$,

$$\int_\Omega u_i\big(g_i(\omega), \xi\big)\, \mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \xi\big)\, \mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\eta\colon \Omega \to A_i$. Therefore, for λ-almost all $i \in I$,

$$\int_\Omega u_i\big(g_i(\omega), \lambda g_\omega^{-1}\big)\, \mathrm{d}\mathbf{P}(\omega) \geq \int_\Omega u_i\big(\eta(\omega), \lambda g_\omega^{-1}\big)\, \mathrm{d}\mathbf{P}(\omega)$$

for every random variable $\eta\colon \Omega \to A_i$. This verifies that g is a mixed-strategy Nash equilibrium of $\mathcal{G}^H$. The proof is now complete. □

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